2022 FALL MATH 5310 HOMEWORK 8 SOLUTIONS
DUE: OCT 24TH

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Question 1 (Artin 4.4.2(a))**.** Let T be a linear operator on a finite-dimensional vector space V, such that T^2 is the identity operator. Prove that for any vector v in V, $v - Tv$ is either an eigenvector with eigenvalue -1, or the zero vector. With notation as in Exercise 4.1, prove that V is the direct sum of the eigenspaces $V^{(1)}$ and $V^{(-1)}$.

Proof. To see a nonzero vector of the form $v - Tv \in V^{(-1)}$, we compute

$$T(v - Tv) = Tv - T^2 v = Tv - v = -(v - Tv).$$

For $V = V^{(1)} \oplus V^{(-1)}$, by definition it suffices to show $V = V^{(1)} + V^{(-1)}$. For this, observe for any $w \in V$, we can decompose it as:

$$w = \frac{w + Tw}{2} + \frac{w - Tw}{2} \in V^{(1)} + V^{(-1)}. \qquad \square$$

(The decomposition foreshadows the real-imaginary decomposition $z = \frac{z+\bar{z}}{2} + \frac{z-\bar{z}}{2}$ where $V \cong \mathbb{C}$ and $T : V \to V$ is the complex conjugation $z \mapsto \bar{z}$.)

Question 2 (Artin 4.4.4)**.** A 2×2 matrix A has an eigenvalue $v_1 = (1, 1)^t$ with eigenvalue 2 and also an eigenvector $v_2 = (1, 2)^t$ with eigenvalue 3. Determine A.

Solution. We have $A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}2\\2\end{bmatrix}$ and $A\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}3\\6\end{bmatrix}$. Combining this we obtain $A\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix} = \begin{bmatrix}2 & 3\\2 & 6\end{bmatrix}$. Therefore,

$$A = \begin{bmatrix}2 & 3\\2 & 6\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}^{-1} = \begin{bmatrix}2 & 3\\2 & 6\end{bmatrix}\begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix} = \begin{bmatrix}1 & 1\\-2 & 4\end{bmatrix}. \qquad //$$

Question 3 (Artin 4.5.1)**.** Compute the characteristic polynomials p and the complex eigenvalues λ and eigenvectors v_λ of

$$\text{(a)} \begin{bmatrix}-2 & 2\\-2 & 3\end{bmatrix} \qquad \text{(b)} \begin{bmatrix}1 & i\\-i & 1\end{bmatrix} \qquad \text{(c)} \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}.$$

Solution. For characteristic polynomials, one can use Proposition 4.5.13 or more specifically Cayley-Hamilton theorem.

(a) Characteristic polynomial $p(t) = t^2 - t - 2$. Eigenvalues: $\lambda = -1, 2$. Eigenvectors: $v_{-1} = \begin{bmatrix}2\\1\end{bmatrix}, v_2 = \begin{bmatrix}1\\2\end{bmatrix}$.

(b) Characteristic polynomial $p(t) = t^2 - 2t$. Eigenvalues: $\lambda = 0, 2$. Eigenvectors: $v_0 = \begin{bmatrix}i\\-1\end{bmatrix}, v_2 = \begin{bmatrix}i\\1\end{bmatrix}$.

(c) Characteristic polynomial $p(t) = t^2 - (2\cos\theta)t + 1$. Eigenvalues: $\lambda = e^{-i\theta}, e^{i\theta}$. Eigenvectors: $v_{e^{-i\theta}} = \begin{bmatrix}i\\1\end{bmatrix}, v_{e^{i\theta}} = \begin{bmatrix}i\\-1\end{bmatrix}$. //

Question 4 (Artin 4.5.2)**.** The characteristic polynomial of the matrix below is $t^3 - 4t - 1$. Determine the missing entries. $\begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 0 \\ 1 & x & y \end{bmatrix}$.

Solution. As there is no t^2-term in the characteristic polynomial, the trace of the matrix is 0. Hence, $y = -1$. Also, as the constant term is -1, we have the determinant is $-(-1) = 1$. Computing the determinant:

$$0 - (-1) + 2(x - 1) = 2x - 1 = 1,$$

so $x = 1$. //

Question 5 (Artin 4.5.3)**.** What complex numbers might be eigenvalues of a linear operator T such that

(a) $T^r = I$,
(b) $T^2 - 5T + 6I = 0$?

Solution. Though the given relations would not determine the characteristic polynomial, we can get possible candidates of eigenvalues from the zeros of the corresponding polynomials: $t^r = 1$ and $t^2 - 5t + 6 = 0$. That is, for (a) we get $t = e^{\frac{2k\pi i}{r}}$ for $0 \le k < r$ as possible eigenvalues of T and, for (b) $t = 2, 3$ are possible eigenvalues of T. //

Question 6 (Artin 4.6.4)**.** Let $A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$. Find a matrix P such that $P^{-1}AP$ is diagonal, and find a formula for the matrix A^{30}.

Solution. The characteristic polynomial of A is $p(t) = t^2 - 4t + 3$, so the eigenvalues are $1, 3$. The corresponding eigenvectors are $v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $v_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Hence, we set the new basis as $\mathcal{B} = \{\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$. Hence, the basis change matrix from the standard one to $\mathcal{B}$ is: $P = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, so we have

$$P^{-1}(AP) = P^{-1}\begin{bmatrix} v_1 & 3v_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.$$

Therefore, $P^{-1}A^{30}P = (P^{-1}AP)^{30} = \begin{bmatrix} 1 & 0 \\ 0 & 3^{30} \end{bmatrix}$, so we obtain

$$A^{30} = P\begin{bmatrix} 1 & 0 \\ 0 & 3^{30} \end{bmatrix}P^{-1} = \frac{1}{2}\begin{bmatrix} 3^{30}+1 & 3^{30}-1 \\ 3^{30}-1 & 3^{30}+1 \end{bmatrix}.$$ //

Question 7 (Bonus; Artin 4.6.8)**.** A linear operator T is nilpotent if some positive power T^k is zero: Prove that T is nilpotent if and only if there is a basis of V such that the matrix of T is upper triangular, with diagonal entries zero.

Proof. Let $T : V^n \to V^n$ be a linear operator. Suppose first T can be represented as an $n \times n$ upper triangular matrix U with diagonal entries zero. Say $P^{-1}TP = U$. Computing the characteristic polynomial of U: $p_U(t) = \det(tI - U) = t^n$, since $tI - U$ is still an upper triangular matrix with all of diagonal entries being t. Hence, we have $p_U(U) = U^n = 0$, so $T^n = P0P^{-1} = 0$, showing that T is nilpotent.

Conversely, suppose T is nilpotent, so $T^k = 0$ for some positive k. We first claim that every eigenvalue of T is zero. Suppose λ is an eigenvalue of T and v is its associated eigenvector. Then $Tv = \lambda v$, so $0 = T^k v = \lambda^k v$, which implies that $\lambda^k = 0$. Hence we prove $\lambda = 0$.

Note that when we triangularize a matrix in Proposition 4.6.1(b), the eigenvalues of the matrix are placed in the diagonal entries of the resulting triangular matrix. Because every eigenvalue of T is zero, T can be triangularized so that all of diagonal entries are zero, which was what was wanted. $\square$