2022 FALL MATH 5310 HOMEWORK 7 SOLUTIONS
DUE: OCT 17TH

SANGHOON KWAK

Question 1 (Artin 3.5.2)**.** Let W_1 be the space of $n \times n$ matrices whose trace is zero. Find a subspace W_2 so that $\mathbb{R}^{n\times n} = W_1 \oplus W_2$.

Solution. One can find $W_2 = \{kI \mid k \in \mathbb{R}\}$. Indeed, for any $A \in \mathbb{R}^{n\times n}$, we have $A = (A - (\operatorname{tr} A)I) + (\operatorname{tr} A)I \in W_1 + W_2$ and $W_1 \cap W_2 = 0$. //

Question 2 (Artin 4.1.4)**.** Prove that every $m \times n$ matrix A of rank 1 has the form $A = XY^t$, where X, Y are m- and n-dimensional column vectors. How uniquely determined are these vectors?

Proof. Let A be an $m \times n$ matrix of rank 1. This means that every column of A can be spanned by a single vector, say v. Then for some $a_1, \ldots, a_n \in \mathbb{R}$, we have $A = \begin{bmatrix} a_1 v & a_2 v & \cdots & a_n v \end{bmatrix} = v \begin{bmatrix} a_1 & \cdots & a_n \end{bmatrix} = XY^t$. Note replacing X with kX and Y with $\frac{1}{k}Y$ for $k \neq 0$ still gives $A = XY^t$, so X and Y are uniquely determined upto scalar multiple. □

Question 3 (Artin 4.2.1)**.** Let A and B be 2×2 matrices. Determine the matrix of the operator $T : M \mapsto AMB$ on the space $F^{2\times 2}$ matrices, with respect to the basis $(e_{11}, e_{12}, e_{21}, e_{22})$ of $F^{2\times 2}$.

Solution. Answer: $\begin{bmatrix} A_{11}B^t & A_{12}B^t \\ A_{21}B^t & A_{22}B^t \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} & A_{11}B_{21} & A_{12}B_{11} & A_{12}B_{21} \\ A_{11}B_{12} & A_{11}B_{22} & A_{12}B_{12} & A_{12}B_{22} \\ A_{21}B_{11} & A_{21}B_{21} & A_{22}B_{11} & A_{22}B_{21} \\ A_{21}B_{12} & A_{21}B_{22} & A_{22}B_{12} & A_{22}B_{22} \end{bmatrix}$. As an illustration, here we see how we get the first column, which records how T maps e_{11} with respect to $(e_{11}, e_{12}, e_{21}, e_{22})$. To see this, we compute:

$$\begin{aligned} Ae_{11}B &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} & A_{11}B_{12} \\ A_{21}B_{11} & A_{21}B_{12} \end{bmatrix} \\ &= A_{11}B_{11}e_{11} + A_{11}B_{12}e_{12} + A_{21}B_{11}e_{21} + A_{21}B_{12}e_{22}. \end{aligned}$$ //

Question 4 (Artin 4.2.3)**.** Find all real 2×2 matrices that carry the line $y = x$ to the line $y = 3x$.

Solution. This amounts to find a matrix T that carries a vector of the form $\frac{1}{k}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ with $k \neq 0$ to a vector $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$. Namely, wrting $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} \frac{1}{k} \\ \frac{1}{k} \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$. This gives $a + b = k$ and $c + d = 3k$. Hence, any matrix of the form

$$T = \begin{bmatrix} a & k - a \\ c & 3k - c \end{bmatrix}$$

with $k \neq 0$ sends $y = x$ to $y = 3x$. //

Question 5 (Artin 4.3.3)**.** Let $T : V \to V$ be a linear operator on a vector space of dimension 2. Assume that T is not multiplication by a scalar. Prove that there is a vector v in V such that $(v, T(v))$ is a basis of V, and describe the matrix of T with respect to that basis.

Solution. Suppose $T\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}k\\0\end{bmatrix}$ and $T\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}0\\\ell\end{bmatrix}$ for some $k, \ell \in \mathbb{R}$. (Otherwise, we can take $\begin{bmatrix}1\\0\end{bmatrix}$ *or* $\begin{bmatrix}0\\1\end{bmatrix}$ as our v.) As T is not a scalar multiple of the identity matrix, we have that $k \neq \ell$. Then taking $v = \begin{bmatrix}1\\1\end{bmatrix}$, we have $Tv = \begin{bmatrix}k\\\ell\end{bmatrix}$, which is not a scalar multiple of v. Hence v, Tv form a basis for $\mathbb{R}^2$.

Now to find the matrix representation A of T, we note:

$$T(v) = Tv$$
$$T^2(v) = av + bTv,$$

for some $a, b \in \mathbb{R}$ since $T^2(v) \in V = \langle v, Tv\rangle$. Therefore, we obtain $A = \begin{bmatrix}0 & a\\1 & b\end{bmatrix}$. (In fact, using Cayley-Hamilton theorem covered later, one can identify $a = -\det T$ and $b = \operatorname{tr} T$.) //

Question 6 (Bonus; Artin 4.M.5)**.** Let $\varphi : F^n \to F^m$ be left multiplication by an $m \times n$ matrix A.

(a) Prove that the following are equivalent:
- A has a right inverse, a matrix B such that $AB = I$,
- φ is surjective,
- the rank of A is m.

(b) Prove that the following are equivalent:
- A has a left inverse, a matrix B such that $BA = I$,
- φ is injective,
- the rank of A is n.

Proof. (a) First assume there exist an $n \times m$ matrix B such that $AB = I$. Then such B represents a linear map $\psi : F^m \to F^n$ such that $\varphi\psi = \mathrm{Id}$. This proves that φ is surjective.

Now assume φ is surjective. Then

$$\operatorname{rk} A = \dim \varphi(F^n) = \dim F^m = m.$$

Finally, assume $\operatorname{rk} A = m$. By the above, this means that $\varphi(F^n) = F^m$, so φ is surjective. Hence φ has a right inverse ψ such that $\varphi\psi = \mathrm{Id}$. Now just setting B to be the matrix representation of ψ, we see B is the right inverse of A.

(b) The argument is symmetric to (a). Having left inverse of A is equivalent to having left inverse of φ, which is equivalent to φ being injective, if and only if $\dim \varphi(F^n) = n$, if and only if $\operatorname{rk} A = n$. □