2022 FALL MATH 5310 HOMEWORK 6 SOLUTIONS
DUE: OCT 3RD

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Question 1 (Artin 3.3.2)**.** Let $W \subset \mathbb{R}^4$ be the space of solutions of the system of linear equations $AX = 0$, where $A = \begin{bmatrix} 2 & 1 & 2 & 3 \\ 1 & 1 & 3 & 0 \end{bmatrix}$. Find a basis for W.

Solution. Reducing the rows, we get the following system:

$$\begin{cases} x - z + 3w = 0 \\ y + 4z - 3w = 0. \end{cases}$$

Solving each equation for x and y respectively we get the following general expression on the vectors in W:

$$(x, y, z, w)^t = (z - 3w, -4z + 3w, z, w)^t = z(1, -4, 1, 0)^t + w(-3, 3, 0, 1)^t,$$

so $\langle(1, -4, 1, 0)^t, (-3, 3, 0, 1)^t\rangle$ is a basis for W. //

Question 2 (Artin 3.3.8)**.** Prove that a set $(v_1, \ldots, v_n)$ of vectors in F^n is a basis if and only if the matrix obtained by assembling the coordinate vectors of v_i is invertible.

Proof. Let A be the matrix obtained by assembling the coordinate vectors of v_i. Then $(v_1, \ldots, v_n)$ forms a basis if and only if the row space of A^t is the whole space F^n, which is equivalent to saying the system $A^tX = 0$ has unique solution $X = 0$, if and only if A^t is invertible, if and only if A is invertible. □

Question 3 (Artin 3.4.1)**.** (a) Prove that the set $B = ((1, 2, 0)^t, (2, 1, 2)^t, (3, 1, 1)^t)$ is a basis of $\mathbb{R}^3$.
(b) Find the coordinate vector of the vector $v = (1, 2, 3)^t$ with respect to this basis.
(c) Let $B' = ((0, 1, 0)^t, (1, 0, 1)^t, (2, 1, 0)^t)$. Determine the basechange matrix P from B to B'.

Proof. (a) Let $P_1 = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$. Reducing P_1^t, we get $\begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -\frac{2}{3} \\ 0 & 0 & 1 \end{bmatrix}$, having no zero rows. This implies that the row space of P_1^t, which is $\langle B\rangle$, is the whole space $\mathbb{R}^3$. This concludes B is a basis.

(b) Note the matrix P_1 above is the basischange matrix from B to the standard one. Because what we want is the other way around, we compute $P_1^{-1} = \frac{1}{7}\begin{bmatrix} -1 & 4 & -1 \\ -2 & 1 & 5 \\ 4 & -2 & -3 \end{bmatrix}$. Therefore, $v = (1, 2, 3)^t$ with respect to B is:

$$P_1^{-1}v = \left(\frac{4}{7}, \frac{15}{7}, -\frac{9}{7}\right)^t.$$

(c) Let $P_2 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ be the basischange matrix from B' to the standard matrix. Then the basischange matrix P from B to B' is nothing but

$$P = P_2^{-1} P_1 = \frac{1}{2} \begin{bmatrix} -1 & 2 & 1 \\ 0 & 0 & 2 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & 1 & 0 \\ 0 & 2 & 1 \\ \frac{1}{2} & 0 & 1 \end{bmatrix}.$$

□

Question 4 (Artin 3.4.2)**.** (a) Determine the basechange matrix in $\mathbb{R}^2$, when the old basis is the standard basis $E = (e_1, e_2)$ and the new basis is $B = (e_1 + e_2, e_1 - e_2)$.

(b) Determine the basechange matrix in $\mathbb{R}^n$, when the old basis is the standard basis E and the new basis is $B = (e_n, e_{n-1}, \dots, e_1)$.

(c) Let B be the basis of $\mathbb{R}^2$ in which $v_1 = e_1$ and v_2 is a vector of unit length making an angle of 120° with v_1. Determine the basechange matrix that relates E to B.

Solution. (a) $P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is the basechange matrix from B to E, so its inverse $P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ is the desired basechange matrix from E to B.

(b) Viewing $e_1, \dots, e_n$ as column vectors, the $n \times n$ matrix $P = \begin{bmatrix} e_n & e_{n-1} & \dots & e_1 \end{bmatrix}$ is the basechange matrix from B to E. Hencee, the desired matrix is its inverse P^{-1}, but one can observe that $P^{-1} = P$. Hence, P is the basechange matrix from E to B.

(c) Since $v_2 = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$, the basechange matrix from E to B is $\begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{\sqrt{3}}{2} \end{bmatrix}^{-1} = \begin{bmatrix} 1 & \frac{\sqrt{3}}{3} \\ 0 & \frac{2}{3}\sqrt{3} \end{bmatrix}$. //

Question 5 (Artin 3.4.3)**.** Let $B = (v_1, \dots, v_n)$ be a basis of a vector space V. Prove that one can get from B to any other basis B' by a finite sequence of steps of the following types:

(i) Replace v_i by $v_i + av_j$, $i \neq j$, for some a in F.
(ii) Replace v_i by cv_i for some $c \neq 0$.
(iii) Interchange v_i and v_j.

Proof. First, note that it suffices to show one can get the standard basis E of $V \cong F^n$ from any basis B using the steps (i–iii), since all of (i–iii) are reversible. Regarding $v_1, \dots, v_n$ as column vectors, we get a matrix $P = \begin{bmatrix} e_1 & e_2 & \dots & e_n \end{bmatrix}$, which is the basischange matrix from B to E. Since P is invertible(Question 2), we can decompose P as a product of elementary column matrices, each of which corresponds to one of the steps (i–iii). This concludes the proof. □

Question 6 (Artin 3.M.2)**.** Let A be a real $n \times n$ matrix. Prove that there is an integer N such that A satisfies a nontrivial polynomial relation $A^N + c_{N-1}A^{N-1} + \ldots + c_1 A + c_0 = 0$.

Proof. The idea is to regard $M_n(\mathbb{R})$, the set of $n \times n$ real matrices as a n^2-dimensional vector space, and to consider the following set of $n^2 + 1$ vectors:

$$S = \{I, A, A^2, \dots, A^{n^2}\} \subset M_n(\mathbb{R}).$$

Since $M_n(\mathbb{R})$ is n^2-dimensional, it follows that S is linearly dependent. Therefore there exist $a_0, \dots, a_{n^2} \in \mathbb{R}$ with $(a_0, \dots, a_{n^2}) \neq (0, \dots, 0)$ such that $\sum_{i=0}^{n^2} a_i A^i = 0$. Say $N \in \{0, \dots, n^2\}$ is the largest index such that $a_N \neq 0$. Note we can further assume $N > 0$. Then simply letting $c_i := a_i / a_N$ for $i = 0, \dots, N-1$, we get

$$A^N + c_{N-1}A^{N-1} + \ldots + c_1 A + c_0 = 0.$$

□