2022 FALL MATH 5310 HOMEWORK 4 SOLUTIONS
DUE: SEP 19TH

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Question 1 (Artin 2.8.6)**.** Let $\varphi : G \to G'$ be a group homomorphism. Suppose that $|G| = 18$, $|G'| = 15$, and that φ is not the trivial homomorphism. What is the order of the kernel?

Solution. Let $k = |\ker\varphi|$. Note $\ker\varphi \leq G$ so $k|18$. From the first isomorphism theorem, we also have $G/\ker\varphi \cong \varphi(G) \leq G'$, so $\frac{18}{k}|15$. Combining those, the only possible values for k are $6, 18$. However, since φ is assumed to be nontrivial, $k \neq 18$ is not valid. Hence, $k = |\ker\varphi| = 6$. //

Question 2 (Artin 2.8.9)**.** Let G be a finite group. Under what circumstances is the map $\varphi : G \to G$ defined by $\varphi(x) = x^2$ an automorphism of G?

Solution. We claim that G is abelian and has odd order. First, as φ is a homomorphism we see that for every $a, b, \in G$:

$$a^2b^2 = \varphi(a)\varphi(b) = \varphi(ab) = (ab)^2 = abab,$$

which implies that $ab = ba$, so G should be abelian.

On the other hand, for φ to be injective, every element has to have odd order. Indeed, say $a \in G$ has order $2m$ for some positive integer m. Then by definition of order, $a^m \neq 1$ and $a^m \in G$ but $\varphi(a^m) = a^{2m} = 1$ breaking the injetivity of φ. Hence, every element of G has odd order. This implies the order of G is odd. (Contrapositive of "Group of even order has order 2 element.") //

Question 3 (Artin 2.11.4)**.** In each of the following cases, determine whether or not G is isomorphic to the product group $H \times K$.

(a) $G = \mathbb{R}^\times$, $H = \{\pm 1\}$, $K = \{\text{positive real numbers}\}$.
(b) $G = \{\text{invertible upper triangular } 2 \times 2 \text{ matrices}\}$, $H = \{\text{invertible diagonal matrices}\}$, $K = \{\text{upper triangular matrices with diagonal entries } 1\}$.
(c) $G = \mathbb{C}^\times$, $H = \{\text{unit circle}\}$, $K = \{\text{positive real numbers}\}$.

Solution. (a) **Yes,** $G \cong H \times K$. The map is $r \mapsto (\frac{r}{|r|}, |r|)$.
(b) **No,** $G \ncong H \times K$. Although the map $(D, U) \mapsto D \cdot U$ gives $G = HK$, one can check that H is not normal in G (that is, a conjugate of a diagonal matrix may not be diagonal.), so failes to be direct product.
(c) **Yes,** $G \cong H \times K$. The map is $x \mapsto (|x|, e^{i \cdot \arg(x)})$ where $|x|$ is the magnitude of x and $\arg(x)$ is the argument of x. //

Question 4 (Artin 2.12.2)**.** In the general linear group $GL_3(\mathbb{R})$, consider the subsets

$$H = \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad K = \begin{bmatrix} 1 & 0 & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

where $*$ represents an arbitrary real number. Show that H is a subgroup of $GL_3(\mathbb{R})$, that K is a normal subgroup of H, and identify the quotient group H/K. Determine the center of H.

Proof. Here we sketch how do we find the quotient group H/K. The key is to come up with a surjective homomorphism from H, with kernel $= K$. One natural construction of such homomorphism is:

$$\varphi : H \longrightarrow (\mathbb{R}^2, +), \qquad \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \longmapsto (a, c) \in \mathbb{R}^2.$$

Here $(\mathbb{R}^2, +)$ is the group $\mathbb{R} \times \mathbb{R}$ equipped with componentwise addition. One can check that this is a surjective homomorphism. Hence, $H/K = H/\ker\varphi \cong \varphi(H) \cong (\mathbb{R}^2, +)$.

To find the center of H, one can observe from computation that

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a' & b' \\ 0 & 1 & c' \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a' & b' \\ 0 & 1 & c' \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{if and only if} \qquad ac' = a'c.$$

Hence, to commute with an arbitrary element $\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \in H$, it follows that $a' = c' = 0$. Therefore $K \subset Z(H)$. The other containtment $K \supset Z(H)$ can be verified by direct computation. □

Question 5 (Artin 2.12.4)**.** Let $H = \{\pm 1, \pm i\}$ be the subgroup of $G = \mathbb{C}^\times$ of fourth root of unity. Describe the cosets of H in G explicitly. Is G/H isomorphic to G?

Solution. Recall $aH = bH$ if and only if $ab^{-1} \in H$. Hence we can describe the cosets of H as:

$$aH = \{\pm a, \pm ai\} = -aH = iaH = -iaH$$

for $a \in G = \mathbb{C}^\times$.

We can realize H as the kernel of the fourth power homomorphism:

$$\varphi : G \longrightarrow G, \qquad z = re^{i\theta} \longmapsto z^4 = r^4 e^{4i\theta},$$

where r is a positive real number and $\theta \in [0, 2\pi)$. Since $r^4 e^{4i\theta}$ still represents every nonzero complex number, φ is surjective. Therefore, the quotient G/H is still isomorphic to G. //

Question 6 (Artin 2.M.14)**.** Prove that the two matrices

$$E = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, E' = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

generate the group $SL_2(\mathbb{Z})$ of all *integer* matrices with determinant 1.

Proof. The key is the *Euclidean algorithm.* Let $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL_2(\mathbb{Z})$. Since $ad - bc = 1$, we have $\gcd(a, c) = 1$. Note E and E' each represents a row operation $R_1 \mapsto R_1 + R_2$ and $R_2 \mapsto R_1 + R_2$. Hence, using powers of E, E' one can perform Euclidean algorithm on the pair (a, c) to make it $(1, 0)$ or $(0, 1)$. After reducing (a, c) to $(1, 0)$ or $(0, 1)$, there are only limited choices for b, d, so the resulting matrix will be one of the forms:

$$\begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix} 0 & -1 \\ 1 & n \end{bmatrix},$$

for some $m, n \in \mathbb{Z}$. Note that $\begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix} = E^m$, and a matrix of the second form can be reduced to the first form by row operations:

$$\begin{bmatrix} 0 & -1 \\ 1 & n \end{bmatrix} \xrightarrow{R_1 \mapsto R_1 + R_2} \begin{bmatrix} 1 & n-1 \\ 1 & n \end{bmatrix} \xrightarrow{R_2 \mapsto R_2 - R_1} \begin{bmatrix} 1 & n-1 \\ 0 & 1 \end{bmatrix},$$

which implies that $E'^{-1}E \begin{bmatrix} 0 & -1 \\ 1 & n \end{bmatrix} = \begin{bmatrix} 1 & n-1 \\ 0 & 1 \end{bmatrix}$.

Therefore, any matrix in $SL_2(\mathbb{Z})$ can be row-reduced using E, E' to one of the forms $\begin{bmatrix} 1 & m \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 1 & n \end{bmatrix}$, which can be further expressed as a product of E, E' and their inverses. This concludes that $SL_2(\mathbb{Z}) = \langle E, E' \rangle$. $\square$