2022 FALL MATH 5310 HOMEWORK 3 SOLUTIONS
DUE: SEP 12TH

SANGHOON KWAK

Question 1 (Artin 2.5.1)**.** Let $\varphi : G \to G'$ be a surjective homomorphism. Prove that if G is cyclic, then G' is cyclic, and if G is abelian, then G' is abelian.

Proof. Suppose G is cyclic, and let $G = \langle g \rangle$ for some $g \in G$. Then $\varphi(g^k) = \varphi(g)^k$ for every $k \in \mathbb{Z}$, so $\varphi(G) = \langle \varphi(g) \rangle$. Because φ is surjective, it follows that $G' = \langle \varphi(g) \rangle$ proving G' is cyclic.

Now suppose G is abelian. Pick arbitrary elements $g', h' \in G'$, and then there exist $g, h \in G$ such that $\varphi(g) = g'$ and $\varphi(h) = h'$ by surjectivity. Then

$$g'h' = \varphi(g)\varphi(h) = \varphi(gh) = \varphi(hg) = \varphi(h)\varphi(g) = h'g',$$

proving G' is abelian. □

Question 2 (Artin 2.5.3)**.** Let U denote the group of invertible triangular 2×2 matrices $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$, and let $\varphi : U \to \mathbb{R}^\times$ be the map that sends $A \mapsto a^2$. Prove that φ is a homomorphism, and determine its kernel and image.

Proof. Note $a \cdot a' = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \begin{bmatrix} a' & b' \\ 0' & d' \end{bmatrix} = \begin{bmatrix} aa' & ab' + bd' \\ 0 & dd' \end{bmatrix}$, so

$$\varphi\left(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\right) \varphi\left(\begin{bmatrix} a' & b' \\ 0' & d' \end{bmatrix}\right) = a^2 a'^2 = \varphi\left(\begin{bmatrix} aa' & ab' + bd' \\ 0 & dd' \end{bmatrix}\right),$$

proving φ is a homomorphism.

The kernel of φ is the set of invertible triangular matrices $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ with $a^2 = 1$, so $a = \pm 1$. Since it is invertible, d should be nonzero. Hence,

$$\ker \varphi = \left\{ \begin{bmatrix} \pm 1 & b \\ 0 & d \end{bmatrix} \middle| \ b, d \in \mathbb{R}, \text{ and } d \neq 0 \right\}.$$

For the image of φ, note that the entry a of A can be any nonzero real number, so a^2 can be all postiive real. Hence, $\varphi(U) = \mathbb{R}_{>0}$, the set of all positive real numbers. □

Question 3 (Artin 2.5.6)**.** Determine the center of $GL_n(\mathbb{R})$. (The homework was to prove this when $n = 2$.)

Solution. When $n = 2$, you could prove the center is the set $\{kI_2 \mid k \in \mathbb{R}\}$ where I_2 is the 2×2 matrix by computing matrix multiplication. Here we give a sketch the idea for general $n \geq 2$. Let $C \in Z(GL_n(\mathbb{R}))$ be a center element. As given in the hint, we use the fact that C commutes with elementary matrices. Note the elementary matrices are multiplied from the left, they give elementary row operations but multiplied from the right, they give elementary *column* operations. Using this, we can narrow down the form of C in the following order:

- By commuting C with an elementary matrix of the first kind(add a multiple of one row/column to another), one can conclude C should be a diagonal matrix, i.e. all the off diagonal entries are zero.
- By commuting C with an elementary matrix of the third kind(multiply a row/column by a nonzero constant), one can conclude all the diagonal entries of C should be the same.

Therefore, we can conclude that $Z(GL_n(\mathbb{R})) \subset \{kI_n \mid k \in \mathbb{R}\backslash\{0\}\}$ where I_n is the $n \times n$ identity matrix. The converse is also true, that all the matrices of the form kI_n are in the center of $GL_n(R)$ because multiplying by kI_n (from either left or right) is just a scalar multiplication by k. This proves that the center is indeed $\{kI_n \mid k \in \mathbb{R} \setminus \{0\}\}$. //

Question 4 (Artin 2.6.1)**.** Let G' be the group of real matrices of the form $\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}$. Is the map $\mathbb{R}^+ \to G'$ that sends x to this matrix an isomorphism?

Proof. Yes, it is an isomorphism. Note $\begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & y \\ & 1 \end{bmatrix} = \begin{bmatrix} 1 & x+y \\ & 1 \end{bmatrix}$, so G' is a multiplicative group. Hence, the map $\varphi : \mathbb{R}^+ \to G'$ defined as $\varphi(x) = \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix}$ is a homomorphism:

$$\varphi(x+y) = \begin{bmatrix} 1 & x+y \\ & 1 \end{bmatrix} = \begin{bmatrix} 1 & x \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & y \\ & 1 \end{bmatrix} = \varphi(x)\varphi(y).$$

Also φ is injective, as $\ker \varphi = 0$, where the identity element of G' is the identity matrix $\begin{bmatrix} 1 & 0 \\ & 1 \end{bmatrix}$. Finally, φ is surjective by definition of G'. This concludes that φ is an isomorphism. □

Question 5 (Artin 2.6.8)**.** Prove that $A \mapsto (A^t)^{-1}$ is an automorphism of $GL_n(\mathbb{R})$.

Proof. First, observe that transposing and taking inverses are commutative on $GL_n(\mathbb{R})$. That is, $(A^t)^{-1} = (A^{-1})^t$. To see why, note for $A \in GL_n(\mathbb{R})$:

$$A^t \cdot (A^{-1})^t = (A^{-1} \cdot A)^t = I_n^t = I_n,$$

where I_n is the $n \times n$ identity matrix. Now we claim that the map $\varphi : GL_n(\mathbb{R})$ defined as $\varphi(A) = (A^t)^{-1}$ is a homomorphism, and it has the inverse as its own, which implies that φ is an automorphism.

First, note

$$\varphi(AB) = ((AB)^t)^{-1} = (B^t A^t)^{-1} = (A^t)^{-1}(B^t)^{-1} = \varphi(A)\varphi(B),$$

so φ is a homomorphism. To show $\varphi^{-1} = \varphi$, we prove $\varphi \circ \varphi = \mathrm{Id}$:

$$(\varphi \circ \varphi)(A) = \varphi((A^t)^{-1}) = (((A^t)^{-1})^t)^{-1} = (((A^t)^t)^{-1})^{-1} = A,$$

where we used the earlier observation that transposing and taking inverses are commuting. All in all, φ is a homomorphism whose inverse, the same as φ, is also a homomorphism. This concludes that φ is an automorphism. □

Question 6 (Bonus question: Artin 2.6.3)**.** Show that the functions $f = 1/x$ and $g = (x-1)/x$ generate a group of functions, the law of composition being composition of functions, that is isomorphic to the symmetric group S_3.

Proof. There are multiple ways to prove this. One way to prove this is that, first observe f has order 2 and g has order 3. Then consider a map that sends f to a transposition $(1,2) \in S_3$ and sends g to a permutation $(1,2,3) \in S_3$. One can check that this map is a homomorphism. It is surjecive as $(1,2)$ and $(1,2,3)$ generate the whole S_3, and then it is automatically injective by noting that the order of $\langle f, g\rangle$ is the same as that of S_3, which is 6.

Here we sketch another idea of proof by analyzing how the group acts on a six point set $X = \{\frac{1}{3}, 3, \frac{2}{3}, \frac{3}{2}, -\frac{1}{2}, -2\}$. Start with $3 \in \mathbb{R}$. After applying f and g to 3 multiple times, one can notice that those multiplications end up six real numbers in X as in the following figure. Note every vertex has valence four; incoming f, g and outgoing f, g.

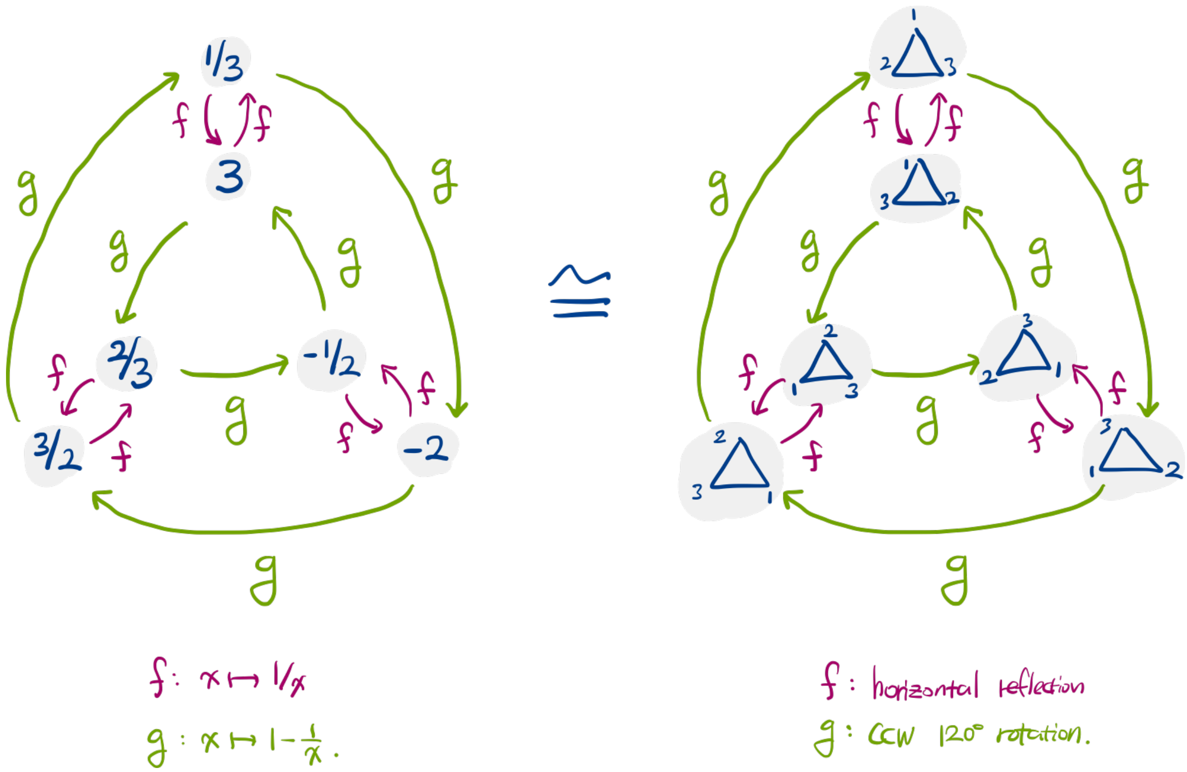

FIGURE 1. Graphical representation of the group $\langle f, g\rangle \cong D_6 \cong S_3$.

Note that this diagram has the same "shape" if you replace each number with properly labeled triangle, where the resulting diagram represents how dihedral group D_6 of order 6 flips and rotates a triangle. Hence, forgetting the role of vertices and edges of each diagram, those two diagrams are completely idential; which implies that the group structures of $\langle f, g\rangle$ and D_6 are idential, so we can conclude $\langle f, g\rangle \cong D_6$, where we know $D_6 \cong S_3$ finishing the proof. If you want to learn more "shapes" of those groups, see e.g. the Wikipedia article of *Cayley Graph.* $\square$