2022 FALL MATH 5310 HOMEWORK 2 SOLUTIONS
DUE: SEP 6TH

SANGHOON KWAK

Question 1. Show that multiplication is well defined in $\mathbb{Z}/m$, i.e. if $x \equiv x'$ and $y \equiv y'$ then $xy \equiv x'y'$.

Proof. Suppose $x \equiv x'$ and $y \equiv y'$ modulo m. Then there exist $a, b \in \mathbb{Z}$ such that $x' = x + am$ and $y' = y + bm$. Hence,

$$\begin{aligned} x'y' - xy &= (x + am)(y + bm) - xy \\ &= xbm + yam + abm^2 = (xb + ya + abm) \cdot m \\ &\equiv 0 \mod m, \end{aligned}$$

which implies that $x'y' \equiv xy$ modulo m. □

Question 2. Let G be a group and X a subset of G. Let $\langle X \rangle$ be the intersection of all subgroups of G that contain X. Show that $\langle X \rangle$ is in fact a subgroup, called *subgroup generated by* X. If $G = GL_n(\mathbb{R})$ and X is the set of all elementary matrices, what subgroup is $\langle X \rangle$?

Proof. First note symbolically we can write

$$\langle X \rangle = \bigcap_{X \subset H \leq G} H.$$

We would like to show $\langle X \rangle$ is a subgroup of G. (Note: In fact, a more general fact holds: *an intersection of collection of subgroups of a group is again a subgroup*. The proof is essentially the same as below.)

- (Identity) $1_G \in \langle X \rangle$ since $1_G = 1_H \in H$ for every subgroup $H \supset X$ of G.
- (Inverse) Also, for any $a \in \langle X \rangle$ we have that $a \in H$ for every $H \supset X$. Since H itself is a subgroup, $a^{-1} \in H$ for every $H \supset X$. Hence, $a^{-1} \in \langle X \rangle$.
- (Closure) Similarly, for any $a, b \in \langle X \rangle$, it follows that $ab \in H$ for every subgroup $H \supset X$, proving $ab \in \langle X \rangle$.

These prove $\langle X \rangle$ is indeed a subgroup of H.

Now set $G = GL_n(\mathbb{R})$, and X to be the set of all elementary matrices. We claim $\langle X \rangle$ is in fact the whole group $GL_n(\mathbb{R})$. We have $\langle X \rangle \subset GL_n(\mathbb{R})$ by definition, so it suffices to prove $\langle X \rangle \supset GL_n(\mathbb{R})$. For this, pick $A \in \mathrm{GL}_n(\mathbb{R})$. Since an invertible matrix is a product of elementary matrices (Theorem 1.2.16), it follows that $A \in \langle X \rangle$, concluding the proof. □

Question 3 (Artin 2.2.4). In which of the following cases is H a subgroup of G?

(a) $G = GL_n(\mathbb{C})$ and $H = \mathrm{GL}_n(\mathbb{R})$.
(b) $G = \mathbb{R}^\times$ and $H = \{1, -1\}$.
(c) $G = \mathbb{Z}^+$ and H is the set of positive integers.
(d) $G = \mathbb{R}^\times$ and H is the set of positive reals.
(e) $G = GL_2(\mathbb{R})$ and H is the set of matrices $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$, with $a \neq 0$.

Solution. (a,b,d) $H \leq G$.
(c) $H \not\leq G$. Note $0 \notin H$.
(e) $H \not\leq G$. Note $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \notin H$. Also $H \not\subset G$ either, as $\det \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} = 0$. //

Question 4 (Artin 2.4.7)**.** Let x and y be elements of a group G. Assume that each of the elements x, y, and xy has order 2. Prove that the set $H = \{1, x, y, xy\}$ is a subgroup of G, and that it has order 4.

Proof. To show $H \leq G$, first note $1 \in H$. Also, it is closed under taking inverses:

$$x^{-1} = x, \qquad y^{-1} = y, \qquad (xy)^{-1} = xy,$$

because x, y and xy are all of order 2. One way to see H is closed under multiplication is to draw the multiplication table(See Section 2.1 of Artin) for H: Here note that since $xyxy = 1$,

	1	x	y	xy
1	1	x	y	xy
x	x	1	xy	y
y	y	yx	1	yxy
xy	xy	xyx	x	1

it follows that $xyx = y^{-1} = y$ and $yxy = x^{-1} = x$. Also $yx = x^{-1}y^{-1} = xy$. Hence after substituting these in the table, H is closed under the multiplication, showing $H \leq G$.

Now to show H is of order 4, it suffices to show $1, x, y$ and xy are distinct. First, x, y and xy are distinct from 1 as they are of order 2. Next, we see x, y are different from xy, otherwise we get $x = 1$ or $y = 1$. Finally, $x \neq y$, otherwise $xy = x^2 = 1$ contradicting the order of xy is 2. Therefore, H has order 4. □

Question 5 (Artin 2.4.9)**.** How many elements of order 2 does the symmetric group S_4 contain?

Solution. The order 2 elements in S_4 are exactly 2-cycles and $(2,2)$-cycles. (See Section 1.5 of Artin.) The number of 2-cycles is $\binom{4}{2} = 6$ and that of $(2,2)$-cycles is $\binom{4}{2}/2 = 3$. Therefore, the total number of order 2 elements is $6 + 3 = 9$. //

Question 6 (Artin 2.9.7)**.** Determine the order of each of the matrices $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ when the matrix entries are interpreted modulo 3.

Solution. One can check that $A^n = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$, so A is of order 3 in $GL_2(\mathbb{Z}/3)$, the set of invertible matrices with entries in $\mathbb{Z}/3$.

On the other hand, $B^n = \begin{bmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{bmatrix}$ where F_n is the n-th term of the *Fibonacci sequence* defined as $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$ and $F_0 = 0, F_1 = 1$. Using this, one can compute that $B^8 = \begin{bmatrix} 34 & 21 \\ 21 & 13 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in GL_2(\mathbb{Z}/3)$, and $B^m \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in GL_2(\mathbb{Z}/3)$ for $m < 8$, proving B is of order 8. //

Question 7 (Bonus question)**.** Construct a group G and elements of $x, y \in G$ of order 2 such that xy has order n for a given integer $n \geq 1$.

Proof. A typical order 2 element to think of is a reflection. The key idea is to observe that *the composition of two parallel reflections is a translation.* (Two non-parallel reflections compose into a rotation.) Using this, we can construct such G. Namely, first consider a real line $\mathbb{R}$. Let x be the reflection at the point $\frac{1}{4}$, and y be the reflection at the point $-\frac{1}{4}$. Then we claim that xy is a translation by $+1$. (Here we follow the function composition notation; first apply y, and then x.) Indeed, for any $r \in \mathbb{R}$, we have

$$xy(r) = x(-\frac{1}{2} - r) = \frac{1}{2} - (-\frac{1}{2} - r) = 1 + r.$$

Hence, if the order of xy is given to be infinity, we can just let x, y as such reflections on $\mathbb{R}$. (In this case $\langle x, y\rangle \cong D_\infty$, called the **infinite dihedral group**.)

However, if n is a finite positive integer, we can make $\mathbb{R}$ into a circle $\mathbb{R}/\sim$ by declaring the equivalence relation $r \sim r + n$ for all $r \in \mathbb{R}$. Then $\mathbb{R}/\sim$ becomes a circle with circumference n, so the n times of $+1$-translation becomes the identity. Now we achived to find the elements x, y of order 2 such that xy is of order n, we declare $G = \langle x, y\rangle$, the group generated by x and y. (See Question 2). In fact, in this case $G \cong D_{2n}$, the dihedral group of order $2n$. $\square$