2022 FALL MATH 5310 HOMEWORK 1 SOLUTIONS
DUE: AUG 29TH

SANGHOON KWAK

Question 1 (Artin 1.1.7)**.** Find a formula for $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^n$ and prove it by induction.

Sketch of proof. Trying $n = 1, 2$, and 3, one can guess that

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n & 1+2+\ldots+n \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & n & \frac{n(n+1)}{2} \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix}.$$

Then we can prove this by induction. □

Question 2 (Artin 1.1.13)**.** Prove that if A is nilpotent, then $I + A$ is invertible.

Proof. Suppose A is nilpotent and $A^k = \mathbf{0}$ for some $k > 0$. Then

$$(I + A)(I - A + A^2 + \ldots + (-1)^{k-1}A^{k-1}) = I + A^k = I,$$

from which we conclude $(I + A)^{-1} = (I - A + A^2 + \ldots + (-1)^{k-1}A^{k-1})$, so invertible. □

Question 3 (Artin 1.2.6)**.** Find the inverse of $\begin{bmatrix} 1&0&0&0&0 \\ 1&1&0&0&0 \\ 1&2&1&0&0 \\ 1&3&3&1&0 \\ 1&4&6&4&1 \end{bmatrix}$.

Solution. One way to find the inverse is to use augmented matrix method (as in Example 1.2.18 in Artin). Namely, start with the following augmented matrix,

$$\left[\begin{array}{ccccc|ccccc} 1&0&0&0&0&1&0&0&0&0 \\ 1&1&0&0&0&0&1&0&0&0 \\ 1&2&1&0&0&0&0&1&0&0 \\ 1&3&3&1&0&0&0&0&1&0 \\ 1&4&6&4&1&0&0&0&0&1 \end{array}\right]$$

and apply row operations until the left side of the augmented matrix becomes the identity matrix:

$$\left[\begin{array}{ccccc|ccccc} 1&0&0&0&0&1&0&0&0&0 \\ 0&1&0&0&0&-1&1&0&0&0 \\ 0&0&1&0&0&1&-2&1&0&0 \\ 0&0&0&1&0&-1&3&-3&1&0 \\ 0&0&0&0&1&1&-4&6&-4&1 \end{array}\right]$$

where the inverse is just the matrix on the right side of the resulting augmented matrix, so

$$\begin{bmatrix} 1&0&0&0&0 \\ 1&1&0&0&0 \\ 1&2&1&0&0 \\ 1&3&3&1&0 \\ 1&4&6&4&1 \end{bmatrix}^{-1} = \begin{bmatrix} 1&0&0&0&0 \\ -1&1&0&0&0 \\ 1&-2&1&0&0 \\ -1&3&-3&1&0 \\ 1&-4&6&-4&1 \end{bmatrix}.$$ //

Question 4 (Artin 1.4.3)**.** Compute the determinant of $\begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & -1 & 2 & -1 & & \\ & & -1 & \ddots & & \\ & & & & \ddots & \\ & & & & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix}$.

Solution. Say the given matrix is M_n. Then using *expansions by minor* in Section 1.6 of Artin, we get a recursive relation on determinants of different n's:

$$\det(M_n) = 2\det(M_{n-1}) - \det(M_{n-2}).$$

Also, by calculating $\det(M_2) = 3$, $\det(M_3) = 4$ and $\det(M_4) = 5$, one can guess $\det(M_n) = n+1$. This can be varified from plugging it in above recursive relation. //

Question 5 (Artin 1.M.7; Vandermonde determinant)**.**

(a) Prove that $\begin{bmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{bmatrix} = (a-b)(b-c)(c-a)$.
(b) Prove an analogous formula for $n \times n$ matrices.
(c) Use the Vandermonde matrix to prove that there is a unique polynomial $p(t)$ of degree n that takes arbitrary prescribed values at $n+1$ points $t_0, \ldots, t_n$.

Solution. (1) One can get the determinant by using *expansion by minor* in Section 1.6 of Artin.

(2) Note applying elementary row operations of Type (i) (as in Section 1.2) does not change the determinant. Hence, we can clear out the first column (except the first entry) by subtracting a_i times i-th row from $(i+1)$-th row for each $i = n-1, n-2, \ldots, 1$:

$$\det\left(\begin{bmatrix} 1 & 1 & \cdots & 1 \\ a_1 & a_2 & \cdots & a_n \\ a_1^2 & a_2^2 & \cdots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{n-1} & a_2^{n-1} & \cdots & a_n^{n-1} \end{bmatrix}\right) = \det\left(\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & a_2 - a_1 & \cdots & a_n \\ 0 & a_2^2 - a_2 a_1 & \cdots & a_n^2 - a_n a_1 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & a_2^{n-1} - a_2^{n-2} a_1 & \cdots & a_n^{n-1} - a_n^{n-2} a_1 \end{bmatrix}\right)$$

$$= \det\left(\begin{bmatrix} a_2 - a_1 & \cdots & a_n - a_1 \\ a_2^2 - a_2 a_1 & \cdots & a_n^2 - a_n a_1 \\ \vdots & \ddots & \vdots \\ a_2^{n-1} - a_2^{n-2} a_1 & \cdots & a_n^{n-1} - a_n^{n-2} a_1 \end{bmatrix}\right)$$

$$= (a_2 - a_1) \cdots (a_n - a_1) \cdot \det\left(\begin{bmatrix} 1 & 1 & \cdots & 1 \\ a_2 & a_3 & \cdots & a_n \\ a_2^2 & a_3^2 & \cdots & a_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ a_2^{n-2} & a_3^{n-2} & \cdots & a_n^{n-2} \end{bmatrix}\right),$$

where the last equality is from factoring each column by $a_2 - a_1, \ldots, a_n - a_1$, respectively. Note the last determinant is exactly the determinant of a Vandermonde matrix with one less variable. Hence, using the same method we can conclude the determinant of $n \times n$ Vandermonde matrix is $\prod_{1 \le i < j \le n} (a_j - a_i)$.

(3) Say $p(x) = a_n x^n + \ldots + a_1 x + a_0$ and $p(t_i) = s_i$ for each $i = 0, \ldots, n$. Then we can form a linear system

$$\begin{bmatrix} 1 & t_0 & \cdots & t_0^n \\ 1 & t_1 & \cdots & t_1^n \\ \vdots & \vdots & \ddots & \vdots \\ 1 & t_n & \cdots & t_n^n \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} s_0 \\ s_1 \\ \vdots \\ s_n \end{bmatrix},$$

where the coefficient matrix is exactly the transpose of $(n+1) \times (n+1)$ Vandermonde matrix, which has nonzero determinant $\prod_{0 \le i < j \le n} (t_j - t_i)$ as $t_0, \ldots, t_n$ are distinct. Hence the above linear system has a unique solution $(a_0, \ldots, a_n)$, giving the unique polynomial p. //

Question 6 (Artin 1.M.11; discrete Dirichlet problem)**.**

(a) Write down the system of linear equations when $R = \{(0,0), (0, \pm 1), (\pm 1, 0)\}$ and solve the Dirichlet problem when β is the function on ∂R defined by $\beta_{uv} = 0$ if $v \le 0$ and $\beta_{uv} = 1$ if $v > 0$.

(b) For discrete harmonic functions, prove the *maximum principle*: a harmonic function takes on its maximal value on the boundary.

(c) Prove that the discrete Dirichlet problem has a unique solution for every region R and every boundary function β.

Solution. (a) Using the given information, a linear system can be formed as

$$\begin{bmatrix} -4 & & & & 1 \\ & -4 & & & 1 \\ & & -4 & & 1 \\ & & & -4 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{bmatrix} \begin{bmatrix} x_{0,1} \\ x_{-1,0} \\ x_{0,-1} \\ x_{1,0} \\ x_{0,0} \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \\ 0 \\ -1 \\ 0 \end{bmatrix}.$$

Solving this (for example using *augmented matrix method* as in Example 1.2.18 in Artin),

one can obtain $\begin{bmatrix} x_{0,1} \\ x_{-1,0} \\ x_{0,-1} \\ x_{1,0} \\ x_{0,0} \end{bmatrix} = \begin{bmatrix} 41/48 \\ 17/48 \\ 5/48 \\ 17/48 \\ 5/12 \end{bmatrix}.$

(b) Since harmonic function on $p \in R$ take the average of four values surrounding p, it is impossible for the interior to take maximal value, whlie bounadry points not. Hence, the maximal values should arise from ∂R. (When f is a constant function, both interior and boundary points take maximal values.)

(c) Let f be a harmonic function. To prove the uniqueness, it suffices to consider when $B = \mathbf{0}$(homogeneous system). This means that the f has 0 on boundary points. By the maximal principle, $f(p) \le 0$ for all $p \in R$, where we claim $f(p) = 0$ for all $p \in R$. Suppose for the sake of contradiction that $f(p) < 0$ for some $p \in R$. A key observation is that $-f$ is also a harmonic function and still have 0 on boundary points. Hence, by the assumption $-f(p) > 0$, which violates the maximum principle, contradiction. Hence, $f(p) = 0$ for all $p \in R$, yielding the unique zero solution for homogeneous system $LX = \mathbf{0}$, proving the uniqueness. //