2022 FALL MATH 5310 HOMEWORK 14 SOLUTIONS
DUE: DEC 7TH

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Question 1 (Artin 7.7.3)**.** How many elements of order 5 might be contained in a group of order 20?

Solution. Let n_5 be the number of Sylow 5-subgroups in G of order 20. Then by the third Sylow theorems, we have $n_5|4$ and $n_5 \equiv 1 \mod 2$, so $n_5 = 1$. Hence there is only one Sylow 5-subgroup $H \leq G$. Because every element of order 5 has to be contained in a Sylow 5-subgroup, which is H of order 5, there can be only four elements of order 5. (Excluding the trivial element from H.) //

Question 2 (Artin 7.7.4(a))**.** Prove that no simple group has order pq, where p and q are prime.

Proof. Let G be a group of order pq. If $p = q$, then G is a p-group. Hence by Proposition 7.3.1, G is either abelian or the center $Z(G)$ is nontrivial proper normal subgroup of G, where both cases imply G is not simple.

Now assume $p \neq q$. Denote by n_p and n_q the number of Sylow p-subgroups and q-subgroups in G. By the third Sylow theorems, we have that

$$n_p|q, \qquad n_p \equiv 1 \mod p,$$
$$n_q|p, \qquad n_q \equiv 1 \mod q.$$

If $n_p = 1$ or $n_q = 1$, then G will have a normal Sylow subgroups. Hence we may assume $n_p = q$ and $n_q = p$. Then $q \equiv 1 \mod p$ and $p \equiv 1 \mod q$. Say $q = pk + 1$ for some $k \geq 1$. Then $p < pk + 1$ so $p \not\equiv 1 \mod (pk + 1)$, contradiction. This concludes that G cannot be simple. □

Question 3 (Artin 7.7.4(b))**.** Prove that no simple group has order p^2q, where p and q are prime.

Proof. When $p = q$, the same proof as in Question 2 shows that the group has to be abelian or has a nontrivial proper center, which is normal. Hence we may assume $p \neq q$.

Let G be a group of order p^2q and denote by n_p and n_q the number of Sylow p- subgroups and Sylow q- subgroups of G. By the third Sylow theorems,

$$n_p|q, \qquad n_p \equiv 1 \mod p,$$
$$n_q|p^2, \qquad n_q \equiv 1 \mod q.$$

Again, if $n_p = 1$ or $n_q = 1$, then G fails to be simple. Hence we may assume $n_p = q$ and $n_q = p$ or $n_q = p^2$. From Question 2, we have seen that $n_q = p$ led to the contradiction, so we assume $n_q = p^2$.

We will derive another contradiction. Say $K_1, \ldots, K_{p^2}$ the Sylow q-subgroups. Since they have prime order q, they have trivial intersections. Hence the number of elements contained in $K_1, \ldots, K_{p^2}$ is $(q-1) \cdot p^2 + 1 = p^2q - p^2 + 1$, where the last $+1$ refers to the trivial element in G, shared by $K_1, \ldots, K_{p^2}$. Because Sylow p-subgroups and Sylow q-subgroups have trivial intersection too, it follows that any Sylow p-subgroup consists of $p^2 - 1$ elements from the elements are not contained in $K_1, \ldots, K_{p^2}$, which are exactly $(p^2 - 1)$-many. This implies that there can be only one Sylow p-subgroup, so $n_p = 1$, contradiction. □

Question 4 (Artin 7.7.5). Find Sylow 2-subgroups of the following groups: (a) D_{10}, (b) T, (c) O, (d) I.

Solution. Denote by n_p the number of Sylow p-subgroups in the group.

(a) Note D_{10} has order 20, and we know $n_5 = 1$, from Question 1. Also we have $n_2|5$ and $n_2 \equiv 1 \mod 2$, so $n_2 = 1$ or $n_2 = 5$. However as D_{10} is non-abelian, we cannot have $n_2 = 1$ as it will give the product structure on D_{10} so that $D_{10} \cong \mathbb{Z}_4 \times \mathbb{Z}_5$ or $D_{10} \cong (\mathbb{Z}_2 \times \mathbb{Z}_2) \times \mathbb{Z}_5$, which are both absurd.

Hence $n_2 = 5$ and $n_5 = 1$. Let $H_1, \ldots, H_5$ be the Sylow 2-subgroups of D_{10}. To find a Sylow 2-subgroup, we can use two order 2 elements that commute: r^5 and s when we identify $D_{10} = \langle r, s \mid r^{10} = s^2 = 1, rs = sr^{-1} \rangle$. Indeed $H_1 = \langle r^5, s \rangle = \{1, r^5, s, r^5 s\}$ has order 4, so a Sylow 2-subgroup. Similarly, we can find other Sylow 2-subgroups by conjugating them:

$$\begin{aligned} H_2 &= \langle r^5, rsr^{-1} \rangle = \langle r^5, r^2 s \rangle \\ H_3 &= \langle r^5, r^2 sr^{-2} \rangle = \langle r^5, r^4 s \rangle \\ H_4 &= \langle r^5, r^3 sr^{-3} \rangle = \langle r^5, r^6 s \rangle \\ H_5 &= \langle r^5, r^4 sr^{-4} \rangle = \langle r^5, r^8 s \rangle. \end{aligned}$$

(b) Note the tetrahedron group T has order 12. Then $n_2|3$ and $n_2 \equiv 1 \mod 2$, so $n_2 = 1$ or $n_2 = 3$. However, we can actually find three order 4 subgroups in T. The idea to find one is to use two order 2 elements that are commute. In T, these are exactly the π-rotations about the center of the edges. Labeling the vertices of the tetrahedron by $1, 2, 3, 4$, these π-rotations can be identified as the three (2,2)-cycles: $a = (1\ 2)(3\ 4)$, $b = (1\ 3)(2\ 4)$, $c = (1\ 4)(2\ 3)$. Note they commute and in particular we have $ab = c$. In other words, the subgroup $H = \langle a, b, c \rangle = \langle a, b \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Also, note H is normal in G as it consists of all $(2, 2)$-cycles in S_4. Therefore, $n_2 = 1$ and H is the only Sylow 2-subgroup of T.

(c) Note the octahedron group O has order 24. Then $n_2|3$ and $n_2 \equiv 1 \mod 2$, so $n_2 = 1$ or $n_2 = 3$. We label the vertices of the octahedron by $0, 1, 2, \ldots, 5$, where 0 is the apex and 5 is the bottom of the octahedron. Here to find an order 8 subgroup of O, we use the $\pi/2$ rotation $r = (1\ 2\ 3\ 4)$ about the axis through 0 and 5, and two π-rotations $a = (1\ 3)(0\ 5)$ and $b(2\ 4)(0\ 5)$ about the axis through 2 and 4, and that through 1 and 3 respectively.

Then we have a relation $ab = ba = c^2$ and $ac = (1\ 4)(2\ 3)(0\ 5) = c^{-1}a$, so $\langle a, b, c \rangle = \{1, a, b, c, c^2, ac^2, bc^2, c^3\}$, a Sylow 8-subgroup of O. Since there are three choices of (unordered) apex–bottom pair for an octahedron, there are two more Sylow 8-subgroups in that form, which are actually conjugate to $\langle a, b, c \rangle$, realizing $n_2 = 3$.

(d) Note the icosahedral group I has order 60 and actually isomorphic to A_5 (Theorem 7.4.4), which is simple (Theorem 7.4.3). From $n_2|15$ and $n_2 \equiv 1 \mod 2$, possible candidates are $n_2 = 3, 5, 15$.

To find an order 4 group of A_5, we can simply use two order 2 elements $a = (1\ 2)(3\ 4)$ and $b = (1\ 3)(2\ 4)$ that are fixing 5. Indeed, $H = \langle a, b \rangle \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, the set of all (2,2)-cycles of A_5, fixing 5. Since we can find such $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup for each stabilizer of one of five points $1, 2, 3, 4, 5$, we have five such Sylow 2-subgroups in total in $I \cong A_5$. //

Question 5 (Artin 7.7.9(a)). Classify groups of order 33.

Proof. Let $|G| = 33$. Denote by n_p the number of Sylow p-subgroups in G. By the third Sylow theorems, $n_3 \equiv 1 \mod 3$ and $n_3|11$. Hence $n_3 = 1$. Similarly, $n_{11} \equiv 1 \mod 11$ and $n_{11}|3$, so $n_{11} = 1$. Say H, K be Sylow 3-subgroup and Sylow 11-subgroup, then $H \cap K = 1$ and both are normal so G is isomorphic to the product $H \times K$, which are just $\mathbb{Z}_3 \times \mathbb{Z}_{11}$ as they have prime order. ∎

Question 6 (Bonus: Artin 7.7.6)**.** Exhibit a subgroup of the symmetric group S_7 that is a nonabelian group of order 21.

Solution. Identify S_7 as the set of all bijective functions $\mathbb{Z}/7 \to \mathbb{Z}/7$. Then consider the function $a_1 : \mathbb{Z}/7 \to \mathbb{Z}/7$ as *adding by 1*, namely $a_1(n) = n + 1$ for all $n \in \mathbb{Z}/7$. Also, consider the function $m_2 : \mathbb{Z}/7 \to \mathbb{Z}/7$ as *multiplying by 2*, namely $m_2(n) = 2n$ for all $n \in \mathbb{Z}/7$. Note a_1 has order 7 and m_2 has order 3 in S_7, but $a_1 m_2 \neq m_2 a_1$, as $2n + 1 \neq 2(n + 1)$. In fact, we have $a_1^2 m_2 = m_2 a_1$.

Now we show $H = \langle a_1, m_2 \rangle$ has order 21. Indeed, using the relation above every element in H can be written in the form $m_2^i a_1^j$ for some $i \in \{0, 1, 2\}$ and $j \in \{0, 1, \ldots, 6\}$. We can check they are all different elements:

$$\begin{array}{lll}
\mathrm{Id} : n \mapsto n, & m_2 : n \mapsto 2n, & m_2^2 : n \mapsto 4n \\
a_1 : n \mapsto n+1, & m_2 a_1 : n \mapsto 2n+2, & m_2^2 a_1 : n \mapsto 4n+4 \\
a_1^2 : n \mapsto n+2, & m_2 a_1^2 : n \mapsto 2n+4, & m_2^2 a_1^2 : n \mapsto 4n+8 = 4n+1 \\
a_1^3 : n \mapsto n+3, & m_2 a_1^3 : n \mapsto 2n+6, & m_2^2 a_1^3 : n \mapsto 4n+12 = 4n+5 \\
a_1^4 : n \mapsto n+4, & m_2 a_1^4 : n \mapsto 2n+8 = 2n+1, & m_2^2 a_1^4 : n \mapsto 4n+16 = 4n+2 \\
a_1^5 : n \mapsto n+5, & m_2 a_1^5 : n \mapsto 2n+10 = 2n+3, & m_2^2 a_1^5 : n \mapsto 4n+20 = 4n+6 \\
a_1^6 : n \mapsto n+6, & m_2 a_1^6 : n \mapsto 2n+12 = 2n+5, & m_2^2 a_1^6 : n \mapsto 4n+24 = 4n+3
\end{array}$$

To show they are different elements, one can use that they have different images for the pair $(0, 1)$. Hence, $H = \langle a_1, m_2 \rangle$ is the desired group. //