2022 FALL MATH 5310 HOMEWORK 13 SOLUTIONS
DUE: NOV 28TH

SANGHOON KWAK

Question 1 (Artin 7.1.1)**.** Does the rule $g * x = xg^{-1}$ define an operation of G on G?

Solution. Yes. It satisfies both axioms for operations: for $x \in G$, $\mathrm{Id} * x = x\,\mathrm{Id}^{-1} = x$ and for $g, h \in G$:

$$(gh) * x = xh^{-1}g^{-1}, \qquad g * (h * x) = g * (xh^{-1}) = xh^{-1}g^{-1}. \qquad //$$

Question 2 (Artin 7.1.2)**.** Let H be a subgroup of a group G. Describe the orbit for the operation of H on G by left multiplication.

Solution. The orbits are the right cosets of H in G; Namely, they are of the form Hg for some $g \in G$. Indeed, say $k \in Hg$. Then $k = hg$ for some $h \in H$, so k is in the same orbit as g under H-action. Conversely, suppose k is in the same orbit as g. Then $k = hg$ for some $h \in H$, so $k \in Hg$. //

Question 3 (Artin 7.2.3)**.** A group G of order 12 contains a conjugacy class of order 4. Prove that the center of G is trivial.

Proof. Let C be a conjugacy class of order 4 in G, and pick $x \in C$. Then by counting formula the centralizer $Z(x)$ of x has order 3. Hence, Since the center Z is a subgroup of $Z(x)$ and $|Z(x)| = 3$, it follows that Z is trivial or $Z = Z(x)$. However the latter cannot happen as x is not centeral; its conjuagcy class is nontrivial. Therefore, $Z = 1$. □

Question 4 (Artin 7.3.2)**.** Let Z be the center of a group G. Prove that if G/Z is a cyclic group, then G is abelian, and therefore $G = Z$.

Proof. Say $G/Z = \langle xZ \rangle = \langle x \rangle Z$ for some $x \in G$. Then for every $a, b \in G$, we can write $a = x^i z$ and $b = x^j w$ for some $z, w \in Z$ and $i, j \in \mathbb{Z}$. Now

$$ab = x^i z x^j w = z x^{i+j} w = w x^{i+j} z = w x^j z x^i = x^j w x^i z = ba,$$

so a and b commute. It follows that G is abelian, so $Z = G$. □

Question 5 (Artin 7.4.2)**.** Is A_5 the only proper normal subgroup of S_5?

Solution. Suppose $H < S_5$ is a proer normal subgroup. Then $H \cap A_5$ is also normal subgroup of S_5, as for every $x \in H \cap A_5$ and $g \in G$, we have $gxg^{-1} \in H$ and $gxg^{-1} \in A_5$ as both H and A_5 are normal subgroups of G. Hence in particular, we have that $H \cap A_5 \trianglelefteq A_5$. However as A_5 is simple, it is either $H \cap A_5 = A_5$ or $H \cap A_5 = 1$. If $H \cap A_5 = A_5$, then we conclude $H = A_5$.

Now, we suppose $H \cap A_5 = 1$. This means that every permutation in H has sign -1 other than the identity element. Because the square of every permutation has sign 1, it follows that $p^2 = \mathrm{Id}$ for every $p \in H$. Thus, the cyclic decomposition of $p \in H$ should only consist of 2-cycles, but cannot contain $(2, 2)$-type such as $(1\ 2)(3\ 4)$ as it is nontrivial positive permutation of sign 1. However as H is normal in S_5, H has to contain all the permutations of the same cyclic decomposition type. As H is closed under multiplication it inevitably has to contain a $(2, 2)$-type permutation, so contradiction.

All in all, the only proper normal subgroup of S_5 is A_5. //

Question 6 (Bonus: Artin 7.2.13)**.** Let N be a normal subgroup of a group G. Suppose that $|N| = 5$ and that $|G|$ is an odd integer. Prove that N is contained in the center of G.

Proof. Using the hint given by Dr. Bestvina, one can consider the homomorphism

$$\varphi : G \to \mathrm{Aut}(N) \cong \mathbb{Z}_4,$$

induced by the conjugation action of G on N. From the first isomorphism theorem $G/\ker(\varphi) \cong \varphi(G) \leq \mathbb{Z}_4$, and that $|G|$ is odd, it follows that $\ker(\varphi) = G$. This means that the conjugation action of G on N is trivial, so N commutes with every element in G. Therefore, we conclude $N \leq Z(G)$.

Alternatively, we can argue as follows. Since $|G|$ is odd and $Z(G) \leq G$, by the class equation every conjugacy class has to have odd order. On the other hand, since N is normal in G, N is a union of such conjugacy classes. From $|N| = 5$ and that N has at least one central element (which is just the identity), either $5 = 1 + 1 + 3$ or $5 = 1 + 1 + 1 + 1 + 1$. Suppose the former, $5 = 1 + 1 + 3$. This means $|Z(G) \cap N| = 2$. However, we know that $Z(G) \cap N$ is a subgroup of G, so $2||G|$, contradiction. Therefore, the latter class equation for N holds, which means that every element of N is central, so $N \leq Z(G)$. □