2022 FALL MATH 5310 HOMEWORK 12 SOLUTIONS
DUE: NOV 21ST

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Question 1 (Artin 6.11.3)**.** Let S be a set on which a group G operates, and let H be the subset of elements g such that $gs = s$ for all s in S. Prove that H is a normal subgroup of G.

Proof. Note that $H = \ker(\varphi : G \to \mathrm{Sym}(S)) \trianglelefteq G$, where φ is the permutation reprentation of the action $G \curvearrowright S$. $\square$

Question 2 (Artin 6.11.4)**.** Let G be the dihedral group D_4 of symmetries of a square. Is the action of G on the vertices a faithful action? on the diagonals?

Solution. The D_4-action on the vertices of a square is faithful, as every nontrivial rotation and reflection moves some vertices to other. However it is *not* faithful on the diagonals, as π-rotation fixes both diagonals. //

Question 3 (Artin 6.11.5)**.** A group G operates faithfully on a set S of five elements, and there are two orbits, one of order 3 and one of order 2. What are the possible groups?

Solution. From the faithful action $G \curvearrowright S$, we have an injective permutation representation $\varphi : G \to S_5$. Also, as G has two orbits as given, we have two restricted permutation representations on each orbit:

$$\varphi_1 : G \to S_2, \qquad \varphi_2 : G \to S_3.$$

Hence, we can combine those to get the following homomorphism:

$$\psi : G \to S_2 \times S_3, \qquad g \mapsto (\varphi_1(g), \varphi_2(g)).$$

Note ψ is injective, as trivial action on both orbits in S is the trivial action on S as a whole, so only the identity can be in the kernel of ψ. Hence, G is isomorphic to a subgroup of $S_2 \times S_3$.

Next, as one orbit has order 2, we can conclude that φ_1 is nontrivial, so φ_1 is surjective. Similarly, we have that the image of φ_2 has to contain a 3-cycle, so the order of $\varphi_2(G)$ is at least 3. Hence, $G \cong \psi(G)$ has $\mathbb{Z}_2$ and $\mathbb{Z}_3$ as subgroups, so $|G|$ is divisible by 6. Among the subgroups of $S_2 \times S_3$, there are only three subgroups whose order is divisible by 6; $S_2 \times S_3$ itself, $S_2 \times \mathbb{Z}_3$, and $1 \times S_3 \cong S_3$. The first two groups do have faithful action on S with the two orbits, each of which is acted by each component of $S_2 \times S_3$ or $S_2 \times \mathbb{Z}_3$. The third one, S_3, also has a faithful action on S, by letting $S = \{1, 2, 3, +, -\}$ where $S_3 \curvearrowright \{1, 2, 3\}$ by permutation, and $S_3 \curvearrowright \{+, -\}$ by the sign of each permutation in S_3. All in all, $G \cong S_2 \times S_3$, or $G \cong S_2 \times \mathbb{Z}_3$, or $G \cong S_3$. //

Question 4 (Artin 6.11.7(a))**.** Find the smallest integer n such that D_4 has a faithful operation on a set of order n.

Solution. Say $|A| = n$ and $D_4 \curvearrowright A$ faithfully. Then the permutation representation $\varphi : D_4 \to S_n$ is injective. Hence, by counting the orders, we have that $8 = |D_4| \leq |S_n| = n!$. Thus, $n \geq 4$.

On the other hand, from Question 2, we have an example of faithful action $D_4 \curvearrowright A$ with $|A| = 4$. That is, just set A to be the set of four vertices of the square associated with D_4. Therefore, $n = 4$ is the smallest number of order of the set on which D_4 faithfully acts. //

Question 5 (Artin 6.11.9)**.** Three sheets of rectangular paper S_1, S_2, S_3 are made into a stack. Let G be the group of all symmetries of this configuration, including symmetries of the individual sheets as well as permutations of the set of sheets. Determine the order of G, and the kernel of the map $G \to S_3$ defined by the permutations of the set $\{S_1, S_2, S_3\}$.

Solution. Each sheet has $2 \times 2 = 4$ symmetries, generated by reflections across x, y-axes when we embed each sheet into xy-plane. Also there are $|S_3| = 3! = 6$ symmetries for shuffling the three sheets. Since the symmetries of each sheet and the shuffling commute and independent, the total number of symmetries is $|G| = 4^3 \times 6 = 384$.

The kernel of the permutation representation $G \to S_{S_1,S_2,S_3} \cong S_3$ is the set of symmetries in G that does not shuffle at all. Hence, the kernel is just $(\mathbb{Z}_2 \times \mathbb{Z}_2)^3 \cong \mathbb{Z}_2^6$, whose order is $2^6 = 64$, generated by the two reflections for each S_1, S_2 and S_3. //

Question 6 (Artin 6.M7(b),(c))**.** Let G be a finite group operating on a finite set S. For each element g of G, let S^g denote the subset of elements of S fixed by $g : S^g = \{s \in S \mid gs = s\}$, and let G_s be the stabilizer of s.

(b) Prove the formula $\sum_{s\in S} |G_s| = \sum_{g\in G} |S^g|$.
(c) Prove *Burnside's Formula:* $|G| \cdot (\text{number of orbits}) = \sum_{g\in G} |S^g|$.

Proof. (b) This is one instance of double counting problem. Enumerate the group elements of G as $g_1, \ldots, g_m$ and the elements of S as $s_1, \ldots, s_k$. Then consider an $m \times k$ table with each entry is either 0 or 1. Each row will be represented by S^{g_i} for some $i = 1, \ldots, n$ and each column of the table will be represented by G_{s_j} for some $j = 1, \ldots, k$. The key observation is the following equivalence:

$$g \in G_s \iff s \in S^g.$$

Hence, for each (i, j)-entry of the table we mark 1 if $g \in G_s$ or equivalently $s \in S^g$, and we mark 0 otherwise. Then $|S^{g_i}|$ will be exactly the sum of i-th row, and $|G^{s_j}|$ will be the sum of j-th column. Therefore, both of the sums

$$\sum_{s\in S} |G_s|, \qquad \sum_{g\in G} |S^g|$$

are the total sum of all the entries of the table, so they are equal.

(c) From the orbit-stabilizer theorem, we know that for each $s \in S$

$$|G| = |G_s| \cdot |G \cdot s|,$$

where $G \cdot s$ is the orbit of s under G. Hence, by replacing $|G_s|$ with $|G|/|G \cdot s|$ from the left hand side of the formula of (b), we get:

$$\sum_{s\in S} |G|/|G \cdot s| = |G| \sum_{s\in S} \frac{1}{|G \cdot s|} = \sum_{g\in G} |S^g|.$$

Finally, observe that for each $s' \in S$ we have $\sum_{s\in G\cdot s'} \frac{1}{|G\cdot s|} = |G \cdot s'| \cdot \frac{1}{|G\cdot s'|} = 1$, so the sum $\sum_{s\in S} \frac{1}{|G\cdot s|}$ counts nothing but the number of orbits of the action $G \curvearrowright S$. All in all, we conclude

$$|G| \cdot (\text{number of orbits}) = \sum_{g\in G} |S^g|.$$ □