2022 FALL MATH 5311 HOMEWORK 11 SOLUTIONS
DUE: NOV 14TH

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Question 1 (Artin 6.7.1)**.** Let $G = D_4$ be the dihedral group of symmetries of the square.

(a) What is the stabilizer of a vertex? of an edge?
(b) G operates on the set of two elements consisting of the diagonal lines. What is the stabilizer of a diagonal?

Solution. (a) Let $D_4 = \langle r, s \mid r^4 = s^2 = 1,\ rs = sr^{-1}\rangle$ where r is the counter clockwise $\frac{\pi}{2}$ rotation, and S is the reflection across a diagonal of the square. Then the stabilizers of vertices will be $\langle s\rangle$ or $\langle sr^2\rangle$ depending on whether the vertex lies on the diagonal across which s is operated. Since $s^2 = (sr^2)^2 = 1$, we conclude the vertex stabilizers are $\langle s\rangle \cong \langle sr^2\rangle \cong \mathbb{Z}/2$.

Similarly, the edge stabilizers are of the form $\langle rs\rangle$ or $\langle r^{-1}s\rangle$, both isomorphic to $\mathbb{Z}/2$.

(b) The diagonal stabilizers are either $\langle s, r^2\rangle$ or $r^{-1}\langle s, r^2\rangle r$, which are both isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$, as $sr^2 = r^2 s$. //

Question 2 (Artin 6.7.7)**.** Let $G = GL_n(\mathbb{R})$ operate on the set $V = \mathbb{R}^n$ by left multiplication.

(a) Describe the decomposition of V into orbits for this operation.
(b) What is the stabilizer of e_1?

Solution. (a) Claim: the orbit decomposition is $\mathbb{R}^n = \{0\} \cup (\mathbb{R}^n \setminus \{0\})$. In fact, for every nonzero vectors $v, w \in \mathbb{R}^n$ one can extend $\{v\}$ and $\{w\}$ to two bases of $\mathbb{R}^n$, where one basis can be mapped to the other by some $A \in GL_n(\mathbb{R})$. In particular, $Av = w$, showing that $GL_n(\mathbb{R})$ is transitive on $\mathbb{R}^n \setminus \{0\}$. On the other hand, $GL_n(\mathbb{R}) \cdot 0 = 0$, so we conclude the decomposition in the aforementioned Claim.

(b) Note for $A \in GL_n(\mathbb{R})$, that Ae_1 is the first column of A. Hence, for $Ae_1 = e_1$ to hold, it follows that the first column of A is e_1. Therefore, the stabilizer of e_1 is $\{A \in GL_n(\mathbb{R}) | \text{The first column of } A \text{ is } e_1\}$, which is indeed a subgroup of $GL_n(\mathbb{R})$. //

Question 3 (Artin 6.7.10(a))**.** Describe the orbit and the stabilizer of the matrix $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ under conjugation in the general linear group $GL_2(\mathbb{R})$.

Solution. The orbit of A under conjugation in $GL_2(\mathbb{R})$ is just the set of matrices that are similar to A, or equivalently the set of 2×2 matrices whose characteristic polynomial is $p(t) = t^2 - 3t + 2$. This is also equivalent to saying the set of 2×2 matrices whose trace is 3 and determinant is 2.

To find the stabilizer of A, one can let $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and compute $P^{-1}AP = A$. Dividing the cases, one obtains $b = c = 0$ so $P = \text{diag}(k, \ell)$ for some $k, \ell \neq 0$, so the stablizer of A is $\{\text{diag}(k, \ell) \mid k, \ell \neq 0\}$. //

Question 4 (Artin 6.9.2)**.** Let G be the group of rotational symmetries of a cube, let G_v, G_e, G_f be the stabilizers of a vertex v, an edge e, and a face f of the cube, and let V, E, F be the sets of vertices, edges, and faces, respectively. Determine the formulas(of the size of orbits) that represent the decomposition of each of the three sets into orbits for each of the subgroups.

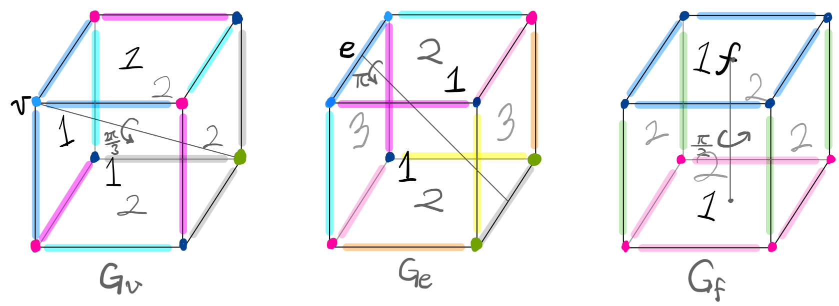

FIGURE 1. Cubes with G_v, G_e, G_f-actions. Vertices(or edges, respectively) in same color are in the same orbit. Faces denoted by the same number in the same orbit.

Solution. Refer to Figure 1. For G_v-action:

- $|V| = 1 + 3 + 3 + 1$.
- $|E| = 3 + 3 + 3 + 3$.
- $|F| = 3 + 3$.

For G_e-action:

- $|V| = 2 + 2 + 2 + 2$.
- $|E| = 1 + 2 + 2 + 2 + 2 + 2 + 1$.
- $|F| = 2 + 2 + 2$

For G_f-action:

- $|V| = 4 + 4$.
- $|E| = 4 + 4 + 4$.
- $|F| = 1 + 4 + 1$.

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Question 5 (Artin 6.9.5)**.** Let F be a section of an I-beam, which one can think of as the product set of the letter I and the unit interval. Identify its group of symmetries, orientation-reversing symmetries included.

Solution. Let $F = I \times [0, 1]$. Then the symmetries of F are generated by the symmetries of I and the symmetries of $[0, 1]$. The nontrivial symmetries of I are the π-rotation r, the vertical reflection s_1, and the horizontal reflection s_2. However, note $r = s_1 s_2$. The only nontrivial symmetry of $[0, 1]$ is the reflection across $\frac{1}{2}$, called $s_{\frac{1}{2}}$. Hence, the symmetry group of F is the group $\langle s_1, s_2, s_{\frac{1}{2}} \rangle$, which in fact is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ as $s_1, s_2, s_{\frac{1}{2}}$ commute. //

Question 6 (Bonus: Artin 6.7.10(b))**.** Interpreting the matrix $A \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ in $GL_2(\mathbb{F}_5)$, find the order of the orbit.

Solution. An easy way to find the order of the orbit is to use orbit-stabilizer theorem. The stabilizer of A under the conjugation in $GL_2(\mathbb{F}_5)$ is the set of 2×2 invertible diagonal matrices as in Question 3. There are $4^2 = 16$ of them. On the other hand, we can compute $|GL_2(\mathbb{F}_5)|$ using a combinatorial argument. Note a matrix is invertible if and only if all of column vectors are linearly independent. Then for the first column of 2×2 invertible matrices in $GL_2(\mathbb{F}_5)$, any nonzero vector can be the first column, so there is $5^2 - 1 = 24$ choice. Given the first column, there are 5 linearly dependent vectors (including the zero vector) to the first column, so there are $5^2 - 5 = 20$ choice for the second column. Hence, $GL_2(\mathbb{F}_5) = 24 \cdot 20 = 480$. All in all, by the orbit-stabilizer theorem, the order of orbit is $480/16 - 30$.

Alternatively, as in Question 3, one can directly find the orbit of A under conjugation in $GL_2(\mathbb{F}_5)$ as the set of matrices whose trace is 3 and determinant is 2. Those are the following 30:

$$\begin{bmatrix} 2 & 0 \\ * & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & * \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 \\ * & 2 \end{bmatrix}, \quad \begin{bmatrix} 1 & * \\ 0 & 2 \end{bmatrix},$$
$$\begin{bmatrix} 3 & 1 \\ 3 & 0 \end{bmatrix}, \quad \begin{bmatrix} 3 & 3 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 3 & 4 \\ 2 & 0 \end{bmatrix}, \quad \begin{bmatrix} 3 & 2 \\ 4 & 0 \end{bmatrix},$$
$$\begin{bmatrix} 0 & 1 \\ 3 & 3 \end{bmatrix}, \quad \begin{bmatrix} 0 & 3 \\ 1 & 3 \end{bmatrix}, \quad \begin{bmatrix} 0 & 4 \\ 2 & 3 \end{bmatrix}, \quad \begin{bmatrix} 0 & 2 \\ 4 & 3 \end{bmatrix},$$
$$\begin{bmatrix} 4 & 3 \\ 3 & 4 \end{bmatrix}, \quad \begin{bmatrix} 4 & 4 \\ 1 & 4 \end{bmatrix}, \quad \begin{bmatrix} 4 & 1 \\ 4 & 4 \end{bmatrix}, \quad \begin{bmatrix} 4 & 2 \\ 2 & 4 \end{bmatrix}.$$

Note there are 18 matrices in total in the first column, as the first and the second have one matrix in common, and the third and the fourth have one matrix in common. //