2022 FALL MATH 5310 HOMEWORK 10 SOLUTIONS
DUE: NOV 7TH

SANGHOON KWAK

Question 1 (Artin 8.5.1(b))**.** Let $V = (\mathbb{R}^n, |\cdot|)$ be the standard Euclidean space. Prove the *parallelogram law* $|v+w|^2 + |v-w|^2 = 2|v|^2 + 2|w|^2$.

Proof.

$$\begin{aligned} |v+w|^2 + |v-w|^2 &= (v+w)\cdot(v+w) + (v-w)\cdot(v-w) \\ &= \left\{|v|^2 + 2v\cdot w + |w|^2\right\} + \left\{|v|^2 - 2v\cdot w + |w|^2\right\} \\ &= 2|v|^2 + 2|w|^2. \end{aligned}$$

□

Question 2 (Artin 8.5.1(c))**.** Let $V = (\mathbb{R}^n, |\cdot|)$ be the standard Euclidean space. Prove that if $|v| = |w|$, then $(v+w) \perp (v-w)$.

Proof. Suppose $|v| = |w|$. Then

$$(v+w)\cdot(v-w) = |v|^2 - |w|^2 = 0,$$

so $(v+w) \perp (v-w)$. □

Question 3 (Artin 8.5.4(a))**.** Let T be a linear operator on $V = \mathbb{R}^n$ whose matrix A is a real symmetric matrix. Prove that V is the orthogonal sum $V = (\ker T) \oplus (\operatorname{im} T)$.

Proof. First, we show $(\operatorname{im} T) \subset (\ker T)^\perp$. Say $v = Tu \in \operatorname{im} T$ for some $u \in V$. Then for any $w \in \ker T$,

$$\langle v, w\rangle = \langle Tu, w\rangle = \langle u, T^t w\rangle = \langle u, Tw\rangle = \langle u, 0\rangle = 0,$$

where we used $T^t = T$ as A is symmetric.

To show $\operatorname{im} T = (\ker T)^\perp$, we count the dimension using rank-nullity theorem:

$$\dim(\operatorname{im} T) = \dim V - \dim(\ker T) = \dim(\ker T)^\perp,$$

proving that $\operatorname{im} T = (\ker T)^\perp$ in fact.

All in all, we conclude that $V = (\ker T) \oplus (\ker T)^\perp = (\ker T) \oplus (\operatorname{im} T)$. □

Question 4 (Artin 8.6.2)**.** Let T be a symmetric operator on a Euclidean space. Using Proposition 8.6.9, prove that if v is a vector and if $T^2 v = 0$, then $Tv = 0$.

Proof. Suppose $T^2 v = 0$. Then $0 = \langle T^2 v, v\rangle = \langle Tv, Tv\rangle = |Tv|^2$, so $Tv = 0$. □

Question 5 (Artin 8.6.3)**.** What does the Spectral Theorem tell us about a real 3×3 matrix that is both symmetric and orthogonal?

Solution. Let A be symmetric and orthogonal. Since A is symmetric, by Spectral theorem there exists an orthogonal matrix P such that $P^t A P$ is a real 3×3 diagonal matrix, say $P^t A P = \operatorname{diag}(a, b, c)$. On the other hand, since A is orthogonal and symmetric, we have $I = A^t A = A^2$. Therefore,

$$\operatorname{diag}(a^2, b^2, c^2) = (P^t A P)^2 = P^t A P P^t A P = P^t A^2 P = P^t P = I,$$

so $a, b, c \in \{\pm 1\}$. All in all, we can conclude that A is diagonalizable with orthogonal matrix, such that the diagonal entries are either 1 or -1. //

Question 6 (Bonus: Artin 8.5.1(a))**.** Let $V = (\mathbb{R}^n, |\cdot|)$ be the standard Euclidean space. Prove the *Schwarz inequality* $|\langle v, w\rangle| \leq |v||w|$.

Proof. Consider $x = u + tv \in V$ for some $u, v \in V$, and $t \in \mathbb{R}$. Then by definition $|x|^2 \geq 0$, so

$$|u + tv|^2 = (u + tv) \cdot (u + tv) = |u|^2 + t^2|v|^2 + 2t(u \cdot v) \geq 0.$$

Rearranging in terms of t, we get the following quadratic inequality:

$$|v|^2t^2 + (2u \cdot v)t + |u|^2 \geq 0.$$

Hence the quadratic function on the left hand side can have at most one zero. This is equivalent to saying that it has a nonpositive discriminant:

$$D = (2u \cdot v)^2 - 4(|v|^2|u|^2) = 4(u \cdot v)^2 - 4|u|^2|v|^2 \leq 0,$$

so we obtain $|u \cdot v| \leq |u||v|$. □