STACK THEORY AND APPLICATIONS

1. Background

1.1. Moduli of triangles. An example of a stack is the moduli stack M of triangles:

An oriented triangle is an ordered triple (x, y, z) of positive real numbers satisfying

$$\begin{aligned} x+y &< z \\ x+z &< y \\ y+z &< x. \end{aligned}$$

There is a universal oriented triangle

$$\tilde{U} \to \tilde{M} \subseteq \mathbb{R}^3_+$$

obtained by going out x units on the positive x-axis, then y units into or through the first quadrant, then closing up. Given a scalene triangle, it has 6 representatives in $\tilde{M}$ corresponding to the $3! = 6$ orderings of the set of distinct numbers $\{x, y, z\}$. However $\{x, x, z\}$ has only three orderings corresponding to the three positions of z and $\{x, x, x\}$ has only one representative in $\tilde{M}$. Thus

$$M = S_3 \backslash \tilde{M}$$

where S_3 denotes the permutation group is a stack, in fact a smooth stack because $\tilde{M}$ is open in $\mathbb{R}^3_+$ and therefore smooth. However there is no universal triangle over M because the action of S_3 on $\tilde{M}$ has isotropy subgroup S_2 at each of the three elements over $\{x, x, z\}$ so we get only half of the isosceles triangle over the corresponding point of M and only one-sixth of the equilateral triangle over $\{x, x, x\}$.

1.2. Fibered categories. To define the kind of stacks we want, we let

$$\mathfrak{S}$$

be the category of schemes of finite type over $Spec\mathbb{C}$. For $X \in Obj(\mathfrak{S})$ we have the contravariant *functor of points*

$$\begin{aligned} h_X &: \mathfrak{S} \to Sets \\ S &\mapsto Hom(S, X) \end{aligned}$$

where a morphism

$$g : S \to T$$

goes to

$$\begin{aligned} Hom(T, X) &\to Hom(S, X). \\ f &\mapsto f \circ g \end{aligned}$$

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Definition 1.1. A functor

$$F : \mathfrak{S} \to Sets$$

is said to be representable if there is $X \in \mathfrak{S}$ such that

$$F \cong h_X.$$

Now each $X \in \mathfrak{S}$ gives rise to its own category which we will denote by

$$\underline{X}.$$

It is given by

$$\begin{aligned} Obj\,(\underline{X}) &= \{(S,u) : S \in \mathfrak{S}, u \in Hom\,(S,X)\} \\ Mor\,(\underline{X}) &= \left\{ \begin{array}{ccccc} S & & \xrightarrow{f} & & T \\ & u \searrow & & \swarrow v & \\ & & X & & \end{array} \right\}. \end{aligned}$$

Then there is a functor

$$\begin{aligned} p &: \underline{X} \to \mathfrak{S} \\ (S,u) &\mapsto S \end{aligned}$$

so that $(\underline{X}, \mathfrak{S})$ is called a *fibered category*. Notice that p_S is itself a category with

(1)
$$\begin{aligned} Obj\,(p_S) &= Hom\,(S,X) \\ Mor\,(p_S) &= \left\{ \begin{array}{ccccc} S & & \xrightarrow{id.} & & S \\ & u \searrow & & \swarrow u & \\ & & X & & \end{array} \right\}. \end{aligned}$$

This is an example of a fibered category which is *fibered by sets*, that is, each category p_S has the property that its only morphisms are the identity maps. In general a category $\mathfrak{C}$ is a fibered category over $\mathfrak{S}$ if it acts like a "sheaf of categories" over $\mathfrak{S}$. That is, given any $S \in Obj\mathfrak{S}$, we assign a category

$$\mathfrak{C}_S \subseteq \mathfrak{C}$$

in such a way that, for any morphism,

$$f : T \to S$$

and any object

$$\xi \in Obj\mathfrak{C}_S$$

there is a well-defined object

$$f^*\xi \in Obj\mathfrak{C}_T$$

and arrow

$$\varphi : f^*\xi \to \xi$$

with the property that, given any $\eta \in Obj\mathfrak{C}_T$ and arrow

$$(\psi : \eta \to \xi) \in Mor\mathfrak{C}$$

there exists a unique

$$(\beta : \eta \to f^*\xi) \in Mor\mathfrak{C}$$

such that

$$\varphi \circ \beta = \psi .$$

(Notice that we do not say that

$$f^* \circ g^* = (g \circ f)^*$$

but some compatibility of the left-hand and right-hand sides of this equation follows from the definition.)

Definition 1.2. A *fibered category* $\mathfrak{C}$ over the category of schemes $\mathfrak{S}$ is a functor

$$p : \mathfrak{C} \to \mathfrak{S}$$

such that

1) every diagram

$$\begin{array}{ccc} & & s \\ & & \downarrow^{p} \\ T & \xrightarrow{F} & p(s) = S \end{array}$$

can be completed to a "cartesian" diagram

$$\begin{array}{ccc} t & \xrightarrow{f} & s \\ \downarrow^{p} & & \downarrow^{p} \\ T = p(t) & \xrightarrow{F=p(f)} & S \end{array} .$$

2) for every diagram

$$\begin{array}{ccccc} & & t' & & \\ & \swarrow_{p} & & \searrow^{h} & \\ T' & & t & \xrightarrow{f} & s \\ & \searrow^{G} & \downarrow^{p} & & \downarrow^{p} \\ & & T & \xrightarrow{F=p(f)} & S \end{array}$$

with

$$p(h) = F \circ G$$

there exists a unique arrow

$$g : t' \to t$$

such that

$$\begin{aligned} G &= p(g) \\ h &= f \circ g . \end{aligned}$$

The question of representability is whether, for a given fibered category

$$\mathfrak{C} \to \mathfrak{S},$$

it is isomorphic to an $\underline{X}$.

Example 1.1.

$$\mathfrak{M}_g \to \mathfrak{S}$$

where we let C_S denote a smooth family of genus-g curves over the scheme S and

$$\begin{aligned} Obj(\mathfrak{M}_g) &= \{C_S\} \\ Mor(\mathfrak{M}_g) &= \{C_S \longleftrightarrow S \times_T C_T\} . \end{aligned}$$

This kind of fibered category is called *fibered in groupoids* of automorphisms of C_S over S.

Now suppose G is an affine algebraic group. We consider the category of G -schemes with

$$Obj\,(G) = \{G_S\}$$

where G_S is an affine group-scheme over S (with "fibers" isomorphic to the product of a fixed affine algebraic group like $GL\,(n)$) and morphisms

$$\{G_S \longleftrightarrow S \times_T G_T\}\,.$$

Example 1.2.

$$BG \to \mathfrak{S}$$

where we fix a group scheme G and let

$$E \to S$$

be a principal G-bundle, that is, we have an action

$$\begin{array}{ccccc} E \times_S G_S & & \longrightarrow & & E \\ & \searrow & & \swarrow & \\ & & S & & \end{array} .$$

We have

$$\begin{array}{rcl} Obj\,(BG) & = & \{E_S\} \\ Mor\,(BG) & = & Hom_{BG}\,(E_S, E'_{S'}) \\ & = & \{E_S \longleftrightarrow S \times_{S'} E'_{S'}\}\,. \end{array}$$

For example

$$BGL\,(n)$$

is the fibered category of rank-n vector bundles over schemes.

Example 1.3. Suppose

$$X \in \mathfrak{S}$$

has a G-action. We might, for example, consider

$$[X/G] \to \mathfrak{S}$$

where

$$Obj\,([X/G])_S = \{f : E_S \to X\}$$

is the set of all G-equivariant maps from principal G-bundles E_S over S, and

$$Mor\,([X/G]) \subseteq Hom_{BG}\,(E_S, E'_{S'})$$

given by

$$\begin{array}{ccccc} E_S & & \longleftrightarrow & & S \times_{S'} E_{S'} \\ & \searrow^{(proj.,f)} & & \swarrow_{1 \times f'} & \\ & & S \times X & & \end{array} .$$

Example 1.4. For example, in Example 1.3 we might take

$$X = point.$$

Then

$$[X/G] = BG.$$

1.3. **2-categories.** A 2-*category* is a category C such that, for any two objects $s, t \in C$,

$$\begin{aligned} Obj_{s,t} &= Mor_C(s,t) \\ Mor_{s,t} &= \left\{ \begin{pmatrix} s & \xrightarrow{f} & t \\ \downarrow & & \downarrow \\ s & \xrightarrow{f'} & t \end{pmatrix} : f, f' \in Mor_C(s,t) \right\} \end{aligned}$$

is a category. The elements of $Mor_{s,t}$ are called 2-*morphisms*. An example of a 2-category is the category of topological spaces and continuous maps in which the two-morphisms are the homotopies of maps.

Referring to the above, given an affine algebraic group G and $X \in \mathfrak{S}$ with a G-action

$$\sigma : G \times X \to X$$

we have a morphism, that is, a functor

$$F : \underline{X} \to [X/G]$$

such that the composition of functors

$$p' \circ F$$

is actually equal to p, not just up to isomorphism. This functor is defined as follows:

$$F(u : S \to X) = \begin{pmatrix} G \times S & \xrightarrow{(1,u)} & G \times X \\ & \searrow & \downarrow \sigma \\ & & X \end{pmatrix}$$

where the G-action on $G \times S$ is via left-multiplication on the first factor.

Another morphism is the morphism

$$\begin{array}{ccccc} BG & & \xrightarrow{F} & & BG' \\ & \searrow p & & \swarrow p' & \\ & & \mathfrak{S} & & \end{array}$$

associated to a homomorphism of affine algebraic groups

$$\varphi : G \to G'.$$

To see what this is, suppose we are given a principal G_S-bundle E_S . We define

$$G' \times^G E := \frac{G' \times E}{\{g' \cdot \varphi(g) \times e \sim g' \times (g \cdot e)\}}$$

that is, the quotient under the $G' \times G$ action on $G' \times E$ under which G acts on the right on G' via φ.

We will later prove:

Proposition 1.1. *$G' \times^G E$ is a principal G'-bundle.*

1.4. **Categories fibered in groupoids.**

Definition 1.3. A fibered category

$$p : \mathfrak{C} \to \mathfrak{S}$$

is *fibered in groupoids* if:

1. For all $s \in Obj(\mathfrak{C})$ and $(F : T \to S = p(s)) \in Mor(\mathfrak{S})$ the set
$$l(F, s) = \{(f : t \to s) \in Mor(\mathfrak{C}) : p(f) = F\}$$
is non-empty. Of course, for
$$s \in l(1_S, s)$$
we have
$$1_s \in l(F, s).$$
We will be interested in other
$$s' \in l(1_S, s).$$
2. The liftings in 1 must satisfy the following condition. Given a commutative triangle
$$\begin{array}{ccc} T' & & \\ & \searrow^{F'} & \\ \downarrow G & & S \\ & \nearrow_{F} & \\ T & & \end{array}$$
in $\mathfrak{S}$, and liftings $f' \in l(F', s)$ and $f \in l(F, s)$, there exists a unique $g \in l(G, t)$ such that
$$\begin{array}{ccc} t' & & \\ & \searrow^{f'} & \\ \downarrow g & & s \\ & \nearrow_{f} & \\ t & & \end{array}$$
commutes.

Notice that, applying 2 to the lifting

$$\begin{array}{ccc} s & & \\ & \searrow^{1_s} & \\ & & s \\ & \nearrow_{f} & \\ s' & & \end{array}$$

of the triangle

$$\begin{array}{ccc} S & & \\ & \searrow^{1_S} & \\ \downarrow 1_S & & S \\ & \nearrow_{1_S} & \\ S & & \end{array}$$

on S one gets that any $f \in l(1_S, s)$ must have a right inverse γ. Applying 2 now to the lifting

$$\begin{array}{ccc} s' & & \\ & \searrow^{1_{s'}} & \\ & & s' \\ & \nearrow_{\gamma} & \\ s & & \end{array}$$

of the identity triangle on S shows that γ has a right inverse δ and so must be an isomorphism and

$$f = f \circ \gamma \circ \delta = \delta$$

is an isomorphism.

So any two t, t' with liftings

$$\begin{array}{ccc} t & & \\ & \searrow & \\ & & s \\ & \nearrow & \\ t' & & \end{array}$$

have to fill in with a unique isomorphism

$$\gamma : t \to t'.$$

So t unique up to unique isomorphism.

Thus, for $S \in Obj(\mathfrak{S})$, if we let

$$\mathfrak{C}_S$$

be the category whose objects are given by

$$p^{-1}(S)$$

and whose arrows are all arrows lying over the identity 1_S, then $\mathfrak{C}_S$ is a *groupoid*, that is, a category all of whose morphisms are left and right invertible.

Recall that any category C has an associated *morphism category* Mor_C with

$$\begin{array}{rcl} Obj(Mor_C) & = & Mor(C) \\ Mor(Mor_C) & = & \left\{ \begin{array}{ccc} c_1 & \to & c_1' \\ \downarrow & & \downarrow \\ c_2 & \to & c_2' \end{array} \right\} \end{array}$$

and that a *natural transformation* n between two functors

$$F, F' : C \to D$$

is a functor

$$n : C \to Mor_D$$

such that

$$n(c) = (F(c) \to F'(c))$$

Given

$$(F : T \to S) \in Mor(\mathfrak{S})$$

and

$$s \in Obj\left(\mathfrak{C}_S\right)$$

we may pick, once and for all, a distinguished element

$$f \in l\left(F, s\right)$$

and call it

$$F_*^s$$

and call its domain

$$F^*\left(s\right).$$

We will call our category fibered in groupoids *special* if the assignment

$$\begin{array}{rcl} Mor\left(\mathfrak{S}\right) \times_{\mathfrak{S}} \mathfrak{C} & \rightarrow & Mor\left(\mathfrak{C}\right) \\ F & \mapsto & F_*^s \end{array}$$

is a functorial in the sense that

$$\left(F \circ G\right)_*^s = F_*^s \circ G_*^{F^*(s)}.$$

(In general in a category fibered in groupoids, we only have a distinguished isomorphism

$$\left(F \circ G\right)_*^s \cong F_*^s \circ G_*^{F^*(s)}$$

not an equality.)

Now for any $s' \in l\left(1_S, s\right)$ we need to compare

$$F^*\left(s\right), F^*\left(s'\right) \in \mathfrak{C}_T$$

We let

$$Iso_S\left(s, s'\right)\left(F\right)$$

be the set of isomorphisms from $F^*\left(s\right)$ to $F^*\left(s'\right)$ over T, that is, the set of morphisms from from $F^*\left(s\right)$ to $F^*\left(s'\right)$ in $\mathfrak{C}_T$. (This may be the empty set.) If we have a commutative diagram

$$\begin{array}{ccc} T & & \\ & \searrow^G & \\ \downarrow H & & X \\ & \nearrow_F & \\ S & & \end{array}$$

then

$$Iso_X\left(x, x'\right)\left(F \circ H\right) \underset{canon.}{\leftrightarrow} Iso_S\left(F^*x, F^*x'\right)\left(H\right)$$

since by property 2 there is a unique pair of isomorphisms α, α' making

$$\begin{array}{ccc} H^*\left(F^*x\right) & \xrightarrow{\alpha} & G^*x \\ \downarrow & & \downarrow \\ H^*\left(F^*x'\right) & \xrightarrow{\alpha'} & G^*x' \end{array}$$

commutative. In what follows our notation will often not distinguish between elements of $Iso_S\left(F^*x, F^*x'\right)\left(H\right)$ and their canonical images in $Iso_X\left(x, x'\right)\left(F \circ H\right)$.

Thus for any $x, x' \in \mathfrak{C}_X := p^{-1}(X)$ we define a functor

$$\begin{array}{rcl} Iso_X(x,x') & : & \underline{X} \to Sets \\ (F : S \to X) & \mapsto & Iso_X(x,x')(F) \end{array}$$

$$\begin{array}{ccccc} T & & & & \\ & \searrow^{G} & & & \\ \downarrow H & & X & \mapsto & \left(\begin{array}{c} Iso_X(x,x')(F) \to Iso_X(x,x')(G) \\ \alpha \mapsto H^*(\alpha) \in Iso_S(F^*x, F^*x')(H) \underset{canon.}{\hookrightarrow} Iso_X(x,x')(G) \end{array} \right). \\ & \nearrow_{F} & & & \\ S & & & & \end{array}$$

A category fibered in groupoids is a fibered category in which the morphisms

$$Mor\mathfrak{C}_S$$

lying over the identity on S are isomorphisms. A special kind of category fibered in groupoids is a *category fibered in sets*, that is, $Obj\mathfrak{C}_S$ is a set and $Mor\mathfrak{C}_S$ consists only in the identity maps on elements of $Obj\mathfrak{C}_S$. In fact, given any functor

$$\mathcal{F} : \mathfrak{S} \to Sets$$

we get a category fibered in sets by putting

$$\begin{array}{rcl} \mathfrak{C} & = & Sets \\ Obj(\mathfrak{C}_S) & = & \{\{x\} : x \in F(S)\} \\ Mor(\mathfrak{C}_S) & = & \left\{ id._{\{x\}} : x \in F(S) \right\}. \end{array}$$

Conversely, given a category fibered in sets, we get a functor

$$\mathcal{F} : \mathfrak{S} \to Sets$$

by noticing that in this case $l(F, s)$ always is a singleton and so we can define

$$\begin{array}{rcl} \mathcal{F}(S) & = & Obj(\mathfrak{C}_S) \\ \mathcal{F}(F : T \to S) & = & \left(\begin{array}{c} Obj(\mathfrak{C}_S) \to Obj(\mathfrak{C}_T) \\ s \mapsto l(F, s) \end{array} \right). \end{array}$$

Example 1.5. The fibered category

$$\mathfrak{C} = \underline{X}$$

is a category fibered in groupoids since

$$F : T \to S,\ S \to X$$

has a unique lifting

$$F^*(s) = s \circ F \in l(s, F)$$

which has no non-trivial conjgations over 1_T. Here

$$F_*^s = \begin{array}{ccc} F^*(s) & \xrightarrow{F} & S \\ \downarrow^{F \circ s} & & \downarrow^{s} \\ X & = & X \end{array}.$$

This category fiered in groupoids is special.

Example 1.6. Suppose that, for the category $\mathfrak{C}$ fibered in groupoids, each

$$\mathfrak{C}_S$$

was a set for each

$$F : T \to S$$

the collection

$$\{f \in Mor(\mathfrak{C}) : p(f) = F\}$$

came from a well-defined (pull-back) map

$$F^* : \mathfrak{C}_S \to \mathfrak{C}_T$$

then there is no ambiguity in defining F^*. Thus over

$$1_S : S \to S$$

we need a fixed morphism

$$\mathfrak{C}_S \to \mathfrak{C}_S.$$

The only thing this can be is the identity map, since the identity map is always an allowable morphism for each object in $\mathfrak{C}$. Also the morphisms between two objects (points) in the category (set) $\mathfrak{C}_S$ must be faithfully represented as maps of a point to a point, so there is at most one. Thus for $s' \in l(s, 1_S)$ and

$$F^*(s), F^*(s') \in \mathfrak{C}_T$$

the set

$$Iso_S(s, s')(F)$$

has exactly one element in it, namely

$$F^*(g)$$

where

$$g : s \to s'$$

is the unique morphism in the category (set) $\mathfrak{C}_S$ taking the point s to the point s'.

If $\mathfrak{C}$ is a presheaf, say for the etale topology, then the pull-back composes correctly.

Proposition 1.2. *A morphism*

$$a : \underline{X} \to \mathfrak{C}$$

is determined up to natural isomorphism by

$$a(X, 1_X) =: A_X.$$

For example, the morphisms

$$\underline{Y} \to \underline{X}$$

are in $1-1$ *correspondence with the morphisms*

$$Y \to X$$

of schemes.

Proof. Given two functors

$$a, b : \underline{X} \to \mathfrak{C}$$

for which $(X, 1_X)$ go to the same object in $\mathfrak{C}$ we obtain a natural isomorphism as follows. For a morphism

$$\begin{array}{ccccc} T & & \xrightarrow{f} & & S \\ & \searrow^{\psi} & & \swarrow_{\varphi} & \\ & & X & & \end{array}$$

we have, by 2 above, a uniquely determined isomorphism

$$\begin{array}{ccccc} a(T, \psi) & \cong & f^* b(S, \varphi) & \to & b(S, \varphi) \\ \downarrow & \swarrow & & & \downarrow \\ T & & \xrightarrow{f} & & S \end{array},$$

so we define the natural transformation via this isomorphism. □

Example 1.7. For the fibered category

$$p : BG \to \mathfrak{S}$$

we have

$$F^*(s) = (E \times_S T \to T) \in l(s, f).$$

We can do the same for morphisms

$$\begin{array}{ccc} x & & \\ & \searrow & \\ \downarrow & & S \\ & \nearrow & \\ y & & \end{array}.$$

The property 2 implies a unique fill in

$$F^* \left(\begin{array}{ccc} x & & \\ & \searrow & \\ \downarrow & & S \\ & \nearrow & \\ y & & \end{array} \right) = \begin{array}{ccc} x' & & \\ & \searrow & \\ \downarrow & & S' \\ & \nearrow & \\ y' & & \end{array} \Longrightarrow \begin{array}{ccc} x & & \\ & \searrow & \\ \downarrow & & S \\ & \nearrow & \\ y & & \end{array}.$$

1.5. **Fibered product of categories fibered in groupoids.** Given a diagram

$$\begin{array}{ccc} \mathfrak{C} & & \\ & \searrow^{f} & \\ & & \mathfrak{E} \\ & \nearrow_{g} & \\ \mathfrak{D} & & \end{array}$$

of categories fibered in groupoids, we define a new category called the *fibered product*:

$$Obj(\mathfrak{C} \times_{\mathfrak{E}} \mathfrak{D}) = \{(x, y, \alpha)\}$$

where

$$\begin{array}{rcl} x & \in & Obj(\mathfrak{C}), y \in Obj(\mathfrak{D}) \\ p_{\mathfrak{C}}(x) & = & p_{\mathfrak{D}}(y) = S \\ \alpha & : & (f(x) \to g(y)) \in l(g(y), 1_S), \end{array}$$

and

$$Mor\left(\mathfrak{C}\times_{\mathfrak{E}}\mathfrak{D}\right)=\left\{F^{*}\left(S\right):\left(x',y',\alpha'\right)\rightarrow\left(x,y,\alpha\right)\right\}$$

where arrows make a big commutative diagram over

$$F:S'\rightarrow S.$$

A bunch of routine diagram checks leads to:

Lemma 1.3. *$\mathfrak{C}\times_{\mathfrak{E}}\mathfrak{D}$ is a category fibered in groupoids.*

A word of caution. The natural diagram

$$\begin{array}{ccccc} & & \mathfrak{C}\times_{\mathfrak{E}}\mathfrak{D} & & \\ & \swarrow & & \searrow & \\ \mathfrak{C} & & & & \mathfrak{D} \\ & \searrow^{f} & & \swarrow_{g} & \\ & & \mathfrak{E} & & \end{array}$$

is only commutative up to isomorphism.

Example 1.8. Consider the category

$$[X/G]$$

of principal G-bundles with G-equivariant maps to X given as in Example 1.3. We have a functor

$$F:\underline{X}\rightarrow[X/G]$$

obtained by associating the trivial G-bundle structure

$$G\times S$$

with G acting on the first factor by left multiplication to S and the morphism

$$\begin{array}{ccc} G\times S & \rightarrow & X \\ (g,s) & \mapsto & g\cdot f\left(s\right) \end{array}$$

to

$$f:S\rightarrow X.$$

We can then form

$$\underline{X}\times_{[X/G]}\underline{X}$$

whose objects are

$$\left(f:S\rightarrow X,f':S\rightarrow X,\left(\begin{array}{ccccc} G\times S & & \overset{\alpha}{\longleftrightarrow} & & G\times S \\ & \searrow^{F(f)} & & \swarrow_{F(f')} & \\ & & X & & \end{array}\right)\right).$$

Notice that α is required to be an isomorphism over S.

The condition on α translates to the statement that an object of the fibered product is (f,f',α) such that

$$\begin{array}{rcl} \alpha\left(g,s\right) & = & \left(g\cdot a\left(s\right),s\right) \\ F\left(f\right) & = & a\left(s\right)\cdot F\left(f'\right) \end{array}$$

So if we set

$$R=G\times X$$

then

$$(a, f') \in Obj\,(\underline{R})$$

gives

$$f = a \cdot f'$$

for which

$$(f, f', \alpha) \in Obj\,\left(\underline{X} \times_{[X/G]} \underline{X}\right).$$

In fact we thereby have an equivalence of categories

$$\underline{X} \times_{[X/G]} \underline{X} \leftrightarrow \underline{G \times X}.$$

Notice that, for a fibered product of categories fibered in groupoids, the choice of f^* on the category

$$\mathfrak{C} \times_{\mathfrak{E}} \mathfrak{D}$$

for

$$\begin{array}{ccc} \mathfrak{C} & & \\ & \searrow^{a} & \\ & & \mathfrak{E} \\ & \nearrow_{b} & \\ \mathfrak{D} & & \end{array}$$

is determined (up to unique isomorphism) by the choices of f^* on $\mathfrak{C}, \mathfrak{D}, \mathfrak{E}$ respectively. The question is what to put for the ? in the diagram

$$\begin{array}{ccc} (f^*x, f^*y, ?) & \rightarrow & (x, y, \alpha) \\ \downarrow & & \downarrow \\ T & \xrightarrow{f} & S \end{array}.$$

One choice is $f^*(\alpha) : f^*(a(x)) \rightarrow f^*(b(y))$ and the other is the required isomorphism $a(f^*(x)) \rightarrow b(f^*(y))$. But these are naturally isomorphic via the the diagram

$$\begin{array}{ccc} a(f^*(x)) & & \\ (\;) & \searrow & \\ f^*(a(x)) & \rightarrow & a(x) \\ \downarrow & & \downarrow \\ T & \xrightarrow{f} & S \end{array}.$$

where property 2 forces a unique element of $l(1_T, f^*(a(x)))$ into the parentheses.

2. Stacks

2.1. **Axioms defining a stack.** In the following definition we have a covering

$$\{T_\alpha \rightarrow S\}$$

in the etale topology, the Grothendieck topology on the category of schemes. We have maps

$$\begin{array}{rcl} i^{\alpha}_{\alpha\beta} & : & T_\alpha \times_S T_\beta \rightarrow T_\alpha \\ i^{\alpha\beta}_{\alpha\beta\gamma} & : & T_\alpha \times_S T_\beta \times_S T_\gamma \rightarrow T_\alpha \times_S T_\beta \\ & & etc. \end{array}$$

If we have a category

$$\mathfrak{C} \to \mathfrak{S}$$

fibered in groupoids, a covering $\{T_\alpha \to S\}$, and a map

$$f : S \to X$$

and, for

$$x, x' \in Obj\,\mathfrak{C}_X$$

an isomorphism

$$s : f^* x \to f^* x'$$

then we have, from the definition of category fibered in groupoids, a commutative diagram of induced distinguished isomorphisms

$$\begin{array}{ccc} i_\alpha^* f^* x & \to & i_\alpha^* f^* x' \\ \downarrow & & \downarrow \\ (f \circ i_\alpha)^* x & \xrightarrow{s_\alpha} & (f \circ i_\alpha)^* x' \end{array}$$

Definition 2.1. A *prestack* is a category

$$\mathfrak{C} \to \mathfrak{S}$$

fibered in groupoids for which "isomorphisms patch," that is, for each $X \in \mathfrak{S}$ and $x, x' \in \mathfrak{C}_X$, the functor

$$\begin{array}{c} I := Iso_X(x, x') : \underline{X} \to Sets \\ (f : S \to X) \mapsto \{f^* x \leftrightarrow f^* x'\} \end{array}$$

is a sheaf. Namely, for a covering

$$\{T_\alpha \to S\}$$

defined over X, the sequence

$$\begin{array}{ccccc} I(S) & \to & \prod_\alpha I(T_\alpha) & \begin{array}{c}\to \\ \to\end{array} & \prod_{\alpha,\beta} I(T_\alpha \times_S T_\beta) \\ s : f^* x \to f^* x' & \mapsto & \{s_\alpha\} & & \\ & & \{s_\alpha\} & \mapsto & \left\{(s_\alpha)_{\alpha\beta}\right\} \end{array}$$

is exact.

Definition 2.2. A *stack* is a prestack for which "descent data is effective," which roughly means that the "functor"

$$\begin{array}{c} \mathfrak{C} : \underline{X} \to Sets \\ (F : S \to X) \mapsto \mathfrak{C}_S \end{array}$$

is a "sheaf" for the etale topology. More precisely, suppose

$$t_\alpha \in \mathfrak{C}_{T_\alpha}.$$

Notice further that an isomorphism

$$\left(\varphi_{\beta\alpha} : \left(i_{\alpha\beta}^\alpha\right)^* t_\alpha \to \left(i_{\alpha\beta}^\beta\right)^* t_\beta\right) \in Mor\left(\mathfrak{C}_{T_\alpha \times_S T_\beta}\right) \tag{2}$$

induces by property 2 of category fibered in groupoids a unique isomorphism

$$\varphi_{\beta\alpha}^{\alpha\beta\gamma} : \left(i_{\alpha\beta\gamma}^{\alpha}\right)^{*} t_{\alpha} \to \left(i_{\alpha\beta\gamma}^{\beta}\right)^{*} t_{\beta}.$$

The property of prestacks that "descent data is effective" is the property that, whenever we have a system of isomorphisms (2) such that

$$\varphi_{\gamma\beta}^{\alpha\beta\gamma} \circ \varphi_{\beta\alpha}^{\alpha\beta\gamma} = \varphi_{\gamma\alpha}^{\alpha\beta\gamma},$$

then there is $t \in \mathfrak{C}_S$ and isomorphisms

$$\left(\psi_{\alpha} : i_{\alpha}^{*}(t) \to t_{\alpha}\right) \in Mor\left(\mathfrak{C}_S\right)$$

such that, uder the induced isomorphisms

$$\left(i_{\alpha\beta}^{\alpha}\right)^{*}\left(i_{\alpha}^{*}(t)\right) \to \left(i_{\alpha\beta}^{\alpha}\right)^{*}\left(t_{\alpha}\right)$$

induced by ψ_{α} we have commutative

$$\begin{array}{ccc} \left(i_{\alpha\beta}^{\alpha}\right)^{*}\left(i_{\alpha}^{*}(t)\right) & \to & \left(i_{\alpha\beta}^{\alpha}\right)^{*}\left(t_{\alpha}\right) \\ \downarrow & & \downarrow \varphi_{\beta\alpha} \\ \left(i_{\alpha\beta}^{\beta}\right)^{*}\left(i_{\beta}^{*}(t)\right) & \to & \left(i_{\alpha\beta}^{\beta}\right)^{*}\left(t_{\beta}\right) \end{array}.$$

(Here the left-hand vertical map can be considered to be the identity under our standard identification

$$\left(i_{\alpha\beta}^{\alpha}\right)^{*}\left(i_{\alpha}^{*}(t)\right) \cong \left(i_{\alpha\beta}\right)^{*}(t) \cong \left(i_{\alpha\beta}^{\beta}\right)^{*}\left(i_{\beta}^{*}(t)\right)$$

via canonical isomorphisms.)

Notice that, if $\mathfrak{C}$ is special, the condition becomes the exactness of the sequence

$$\mathfrak{C}_S \to \prod_{\alpha} \mathfrak{C}_{T_{\alpha}} \begin{array}{c}\to\\\to\end{array} \prod_{\alpha,\beta}\left(\mathfrak{C}_{T_{\alpha} \times_S T_{\beta}}\right).$$

In the case that $\mathfrak{C}$ is fibered in sets, this just becomes the condition that

$$\begin{array}{ccc} \mathfrak{C} & : & \mathfrak{S} \to Sets \\ S & \mapsto & \mathfrak{C}_S \end{array}$$

is a sheaf.

2.2. **Representability.** The notion of representability is a bit different for a morphism (functor) of stacks

$$\varphi : \mathfrak{F}' \to \mathfrak{F} \tag{3}$$

over $\mathfrak{S}$.

Definition 2.3. The morphism (3) of stacks fibered in groupoids is (strongly) representable if, for any scheme X and any morphism (functor)

$$f : \underline{X} \to \mathfrak{F},$$

the fibered product

$$\underline{X} \times_{\mathfrak{F}} \mathfrak{F}'$$

is isomorphic to a category

$$\underline{X_{\varphi,f}}$$

for some

$$X_{\varphi,f} \in Obj\,(\mathfrak{S})\,.$$

Later we will have a notion of weak representability in which X and X_φ are algebraic spaces rather than schemes.

Referring to 1.5 recall that representability of the fibered product requires for each $f : \underline{X} \to \mathfrak{F}$:

1. A morphism
$$f_\# : X_{\varphi,f} \to X$$
of schemes.
2. An equivalence (that is, a fully faithful essentially surjective functor) between the category of triples consisting in
$$y : Y \to X \in Mor\,(\mathfrak{S})\,,$$
an object
$$A_Y \in \mathfrak{F}'_Y\,,$$
and an isomorphism
$$\begin{array}{ccc} \varphi\,(A_Y) & \in & \mathfrak{F}_Y \\ \updownarrow \alpha & & \\ f\,(y) & \in & \mathfrak{F}_Y \end{array}\,,$$
and the category of factorizations
$$Y \to X_{\varphi,f} \xrightarrow{f_\#} X.$$

Example 2.1. The identity functor on $\mathfrak{F}$ is always representable. For each $f : \underline{X} \to \mathfrak{F}$, define

$$\begin{array}{rcl} X_{id.} & = & X \\ f_\# & = & id. \end{array}$$

Then the equivalence of categories sends

$$y : Y \to X$$

to the triple

$$y : Y \to X,\ f\,(y) \in \mathfrak{F}_Y,\ 1_{f(y)}$$

Definition 2.4. A stack $\mathfrak{F}$ is representable if

$$\underline{X} \cong \mathfrak{F}$$

is an equivalence of categories for some scheme X.

If P is a property of morphisms

$$S \to T$$

of schemes which is local over T and preserved under base extension, then a representable morphism (3) of stacks over schemes inherits property P if all

$$f_\# : X_{m,f} \to X$$

have property P.

Definition 2.5. A stack $\mathfrak{F}$ over $\mathfrak{S}$ is called *algebraic* if the diagonal morphism

$$\mathfrak{F} \to \mathfrak{F} \times_{\mathfrak{S}} \mathfrak{F}$$

is representable and quasi-compact.

3. Descent

3.1. **Covers in the Grothendieck topology.** To motivate this concept, let X be a scheme and

$$i_\alpha : U_\alpha \to X$$

the inclusions making up a Zariski open cover.

Proposition 3.1. 1. *For a scheme S and a collection of morphisms*

$$f_\alpha : U_\alpha \to S$$

such that, for the fibered product (intersection)

$$\begin{array}{ccc} U_\alpha \times_X U_\beta & \xrightarrow{i^\beta_{\alpha,\beta}} & U_\beta \\ \downarrow i^\alpha_{\alpha\beta} & & \downarrow i_\beta \\ U_\alpha & \xrightarrow{i_\alpha} & X \end{array},$$

we have

$$f_\alpha \circ i^\alpha_{\alpha\beta} = f_\beta \circ i^\beta_{\alpha,\beta},$$

then there exists a unique

$$f : X \to S$$

such that

$$f_\alpha = f \circ i_\alpha.$$

2. *If F and G are quasi-coherent sheaves on X and we put $F_\alpha = i^*_\alpha F$, etc., and we have*

$$g_\alpha : F_\alpha \to G_\alpha$$

for which

$$\left(i^\alpha_{\alpha\beta}\right)^* g_\alpha = \left(i^\beta_{\alpha\beta}\right)^* g_\beta,$$

then there exists a unique

$$g : F \to G$$

for which

$$g_\alpha = i^*_\alpha g.$$

3. *Given a collection F_α of quasi-coherent sheaves on U_α and isomorphisms*

$$\varphi_{\alpha\beta} : \left(i^\beta_{\alpha\beta}\right)^* F_\beta \to \left(i^\alpha_{\alpha\beta}\right)^* F_\alpha$$

for which

$$\left(i^{\alpha\gamma}_{\alpha\beta\gamma}\right)^* \varphi_{\alpha\gamma} = \left(i^{\alpha\beta}_{\alpha\beta\gamma}\right)^* \varphi_{\alpha\beta} \circ \left(i^{\beta\gamma}_{\alpha\beta\gamma}\right)^* \varphi_{\beta\gamma}$$

then there exists a quasi-coherent sheaf F and isomorphisms

$$\psi_\alpha : i^*_\alpha F \to F_\alpha$$

such that

$$\varphi_{\alpha\beta} = \left(\left(i_{\alpha\beta}^{\alpha}\right)^{*}\psi_{\alpha}\right) \circ \left(\left(i_{\alpha\beta}^{\beta}\right)^{*}\psi_{\beta}^{-1}\right).$$

To see the significance of the above reformulation of the above elementary lemma in sheaf theory, form a category

$$Qcoh\left(\{U_{\alpha}\}\right)$$

whose objects are collections

$$\left(F_{\alpha}, \left\{\varphi_{\alpha\beta} : \left(i_{\alpha\beta}^{\beta}\right)^{*} F_{\beta} \to \left(i_{\alpha\beta}^{\alpha}\right)^{*} F_{\alpha}\right\}_{\beta}\right)$$

as in 3 and whose morphisms are data

$$\left(g_{\alpha} : F_{\alpha} \to G_{\alpha}\right)$$

as in 2 which "commute with the descent data," that is

$$\varphi_{\alpha,\beta}^{G} \circ \left(i_{\alpha\beta}^{\beta}\right)^{*} g_{\beta} = \left(i_{\alpha\beta}^{\alpha}\right)^{*} g_{\alpha} \circ \varphi_{\alpha,\beta}^{F}.$$

There is an obvious functor

$$\{i_{\alpha}^{*}\} : Qcoh\left(X\right) \to Qcoh\left(\{U_{\alpha}\}\right)$$

and the property 2 is the assertion that this functor is fully faithful whereas the property 3 is the assertion that this functor is essentially surjective. Thus the two properties together prove:

Theorem 3.2. *$\{i_{\alpha}^{*}\}$ is an equivalence of categories.*

Descent is nothing more than the corresponding assertion of an equivalence of categories in the case in which, in the entire discussion above, we replace the Zariski topology with the Grothendieck topology for faithfully flat morphisms of finite presentation or for etale morphisms. Thus in the above Proposition and its consequences the maps i_{α} become flat morphisms π_{α} of finite presentation or etale morphisms and all else is unchanged.

Example 3.1. Let G be an affine algebraic group and let

$$\pi : E \to X$$

be a principal G-bundles. Now the statement that E is a principal G -bundle is the statement that π is flat and that the diagram

$$\begin{array}{ccc} E \times G & \xrightarrow{\alpha} & E \\ \downarrow p_{E} & & \downarrow \pi \\ E & \xrightarrow{\pi} & X \end{array}$$

is cartesian, where

$$\alpha\left(e, g\right) = e \cdot g$$

is the G-action on E. Thus

$$(4) \qquad \begin{array}{rcl} \left(p_{E}, \alpha\right) & : & E \times G \cong E \times_{X} E \\ \left(e, g, h\right) & \mapsto & \left(e, e \cdot g\right) \end{array}$$

and

$$
\begin{array}{rcl}
E \times G \times G & \cong & E \times_X E \times G \cong E \times_X E \times_X E \\
(e,g,h) & \mapsto & (e, e \cdot g, h) \mapsto (e, e \cdot g, e \cdot gh)
\end{array} \tag{5}
$$

Example 3.2. There isomorphism allow us to write descent data for the flat "covering" $\{\pi\}$ consisting of a single covering map. We then have for the fibered product of three covering maps

$$
\begin{array}{ccccccccccc}
 & & & & & E \times G \times G & & & & & \\
 & & & \swarrow_{p_{12}} & & \downarrow_{p_{13}} & & \searrow^{p_{23}} & & & \\
 & & E \times G & & & E \times G & & & E \times G & & \\
 & \swarrow_{p_E} & {}^{p_E}\swarrow & \searrow^{\alpha} & & & \swarrow_{p_E} & & \searrow^{\alpha} & \searrow^{\alpha} & \\
E & & & & & E & & & & & E
\end{array}
$$

where, in (5),

$$
\begin{array}{rcl}
p_{12}\,(e,g,h) & = & (e,g) \\
p_{13}\,(e,g,h) & = & (e,gh) \\
p_{23}\,(e,g,h) & = & (eg,h)
\end{array}
$$

and we denote

$$\pi_1 = p_E \circ p_{12} = p_E \circ p_{13}$$

For example, vector bundles on X can be thought of as given by their descent data, namely, a quasi-coherent sheaf F on the domain of the only covering map π together with an isomorphism

$$\varphi : \alpha^* F \to p_E^* F$$

Definition 3.1. A G-equivariant quasi-coherent sheaf on E is a quasi-coherent sheaf F on E and an isomorphism

$$\varphi : \alpha^* F \to p_E^* F$$

Thus by Theorem 3.2 we have:

Theorem 3.3. *There is an equivalence of categories between the category of quasi-coherent sheaves on X (for the chosen Grothendieck topology) and the category of G-equivariant quasi-coherent sheaves on E.*

Proof. The isomorphism φ corresponds to the cartesian diagram

$$
\begin{array}{ccc}
F \times G & \xrightarrow{\alpha'} & F \\
\downarrow & & \downarrow \\
E \times G & \xrightarrow{\alpha} & E
\end{array}
$$

where we write

$$\alpha'\,(v,g) = v \cdot g$$

□

Exercise 3.1. The cocycle condition 3 on φ in Proposition 3.1 above is equivalent to the condition that α' is an action.

Proof. The cocyle condition says

$$p_{23}^*(\varphi) \circ p_{12}^*(\varphi) = p_{13}^*(\varphi) : F \times G \times G \to F \times G \times G$$

and comes from lifting

$$\varphi : \alpha^* F \to p_E^* F$$

under the commutative cube

$$\begin{array}{ccccccc}
 & & \underset{(e,g,h)}{E \times G \times G} & - \quad p_{23} \quad \rightarrow & & & \underset{(eg,h)}{E \times G} \\
 & \swarrow p_{12} & & & & \swarrow p_E & \\
\underset{(e,g)}{E \times G} & & - & \alpha \quad \rightarrow & \underset{eg}{E} & & | \\
 & & p_{13} & & & & \alpha \\
 | & & \downarrow & & | & & \downarrow \\
 p_E & & & & \pi & & \\
 \downarrow & & \underset{(e,gh)}{E \times G} & - \quad \alpha & \downarrow & \rightarrow & \underset{egh}{E} \\
 & \swarrow p_E & & & & \swarrow \pi & \\
 \overset{e}{E} & & - & \pi \quad \rightarrow & X & &
\end{array}$$

to obtain a commutative system of isomorphisms from

$$\begin{array}{ccccccc}
 & & \pi_1^* F & - \quad (p_{23})_\# \quad \rightarrow & & & p_E^*(F) \\
 & \swarrow (p_{12})_\# & & & & \swarrow (p_E)_\# & \\
p_E^*(F) & & - & \alpha_\# \quad \rightarrow & F & & | \\
 & & (p_{13})_\# & & & & \alpha_\# \\
 | & & \downarrow & & | & & \downarrow \\
 (p_E)_\# & & & & \pi & & \\
 \downarrow & & p_E^*(F) & - \quad \alpha_\# \quad \rightarrow & & & F \\
 & \swarrow (p_E)_\# & & & & & \\
 F & & & & & &
\end{array}$$

to

$$\begin{array}{ccccccc}
 & & \underset{(f,g,h)}{F \times G \times G} & - \quad p_{23} \quad \rightarrow & & & \underset{(fg,h)}{F \times G} \\
 & \swarrow p_{12} & & & & \swarrow p_F & \\
\underset{(f,g)}{F \times G} & & - & \alpha' \quad \rightarrow & \underset{fg}{F} & & | \\
 & & p_{13} & & & & \alpha' \\
 | & & \downarrow & & & & \downarrow \quad . \\
 p_F & & & & & & \\
 \downarrow & & \underset{(f,gh)}{F \times G} & - \quad \alpha' \quad \rightarrow & & & \underset{fgh}{F} \\
 & \swarrow p_E & & & & & \\
 \overset{f}{F} & & & & & &
\end{array}$$

□

3.2. **Single morphism covers.**

Definition 3.2. A morphism of schemes

$$f : T' \to T$$

is *quasi-compact* if the preimage of every affine open is a finite union of affine opens.

We can simplify the notation for descent data when we restrict ourselves to covering consisting of a single faithfully flat, quasi-compact surjective map

$$f : T' \to T.$$

(Recall that *faithfully flat* means that $\mathcal{O}_{T'}$ is faithfully flat over $f^{-1}\mathcal{O}_T$, that is,

$$\mathcal{O}_{T'} \otimes_{\mathcal{O}_T}$$

is fully faithful, that is, exactness of a sequence after tensoring is equivalent to exatness before tensoring.) We have

$$\begin{array}{ccccccccccc} & & & & & T' \times_T T' \times_T T' & & & & & \\ & & & \swarrow_{p_{12}} & & \downarrow_{p_{13}} & & \searrow^{p_{23}} & & & \\ & & T' \times_T T' & & & T' \times_T T' & & & T' \times_T T' & & \\ & \swarrow_{p_1} & \swarrow_{p_1} & \searrow^{p_2} & & & & \swarrow_{p_1} & \searrow^{p_2} & \searrow^{p_2} & \\ T' & & & & & T' & & & & & T' \\ & & \searrow_f & & & \downarrow_f & & & \swarrow_f & & \\ & & & & & T & & & & & \end{array}$$

or finally simply

$$\begin{array}{c} T' \times_T T' \times_T T' \\ \downarrow_{p_{12}} \downarrow_{p_{13}} \downarrow_{p_{23}} \\ T' \times_T T' \\ \downarrow_{p_1} \downarrow_{p_2} \\ T' \\ \downarrow_f \\ T \end{array}$$

where we denote

$$\begin{array}{rcl} \pi_1 & : & = p_1 \circ p_{12} = p_1 \circ p_{13} \\ \pi_2 & : & = p_2 \circ p_{12} = p_1 \circ p_{23} \\ \pi_3 & : & = p_2 \circ p_{13} = p_2 \circ p_{23}. \end{array}$$

We define a category

$$\mathfrak{C}^{descent}_{T' \to T}$$

with

$$\begin{array}{rcl} Obj\left(\mathfrak{C}^{descent}_{T' \to T}\right) & = & \{(u \in \mathfrak{C}_{T'},\ \varphi : p_1^* u \cong p_2^* u) : p_{23}^* \varphi \circ p_{12}^* \varphi = p_{13}^* \varphi\} \\ Mor\left(\left(\mathfrak{C}^{descent}_{T' \to T}\right)\right) & = & \left\{ h : u \to v \in Mor\left(\mathfrak{C}_{T'}\right) : \begin{array}{ccc} p_1^* u & \xrightarrow{\varphi} & p_2^* u \\ \downarrow p_1^* h & @ & \downarrow p_2^* h \\ p_1^* v & \xrightarrow{\psi} & p_2^* v \end{array} \right\} \end{array}$$

and a descent functor

$$\begin{array}{rcl} df & : & \mathfrak{C}_T \to \mathfrak{C}^{descent}_{T' \to T}. \\ u & \mapsto & (f^* u, \varphi : p_1^* \circ f^* u \to p_1^* \circ f^* u) \end{array}$$

We then have the following alternative definition:

Definition 3.3. A category fibered in groupoids

$$\mathfrak{C} \to \mathfrak{S}$$

is a stack if, for all etale surjective maps

$$f : T' \to T$$

of locally finite presentation over T, df is an equivalence of categories.

It is then a theorem that, if *df* an equivalence of categories for f etale, locally finitely presented over T, then it is also an equivalence of categories for f faithfully flat of locally finite presentation over T .

3.3. **Descent and quasi-coherent sheaves.**

1. For a quasi-coherent F' sheaf on T' together with an isomorphism
$$\varphi : p_1^* F' \to p_2^* F'$$
satisfying the cocycle condition
$$p_{23}^* \varphi \circ p_{12}^* \varphi = p_{13}^* \varphi,$$
then F' descends to a sheaf F on T together with an isomorphism
$$\vartheta : F' \to f^* F$$
such that the diagram
$$\begin{array}{ccccc} p_1^* F' & & \xrightarrow{\varphi} & & p_2^* F' \\ & \searrow^{p_1^*} & & \swarrow_{p_2^*} & \\ & & F' & & \end{array}$$
is commutative. Thus the datum of a quasi-coherent sheaf on T is equivalent to descent data.
2. Given descent data
$$(F', \varphi), (G', \psi)$$
then the datum of a morphism
$$\alpha : F \to G$$
is equivalent to the morphism descent data consisting of
$$\alpha' : F' \to G'$$
and a commutative diagram
$$\begin{array}{ccc} p_1^* F' & \xrightarrow{\varphi} & p_2^* F' \\ \downarrow_{p_1^* \alpha'} & & \downarrow_{p_2^* \alpha'} \\ p_1^* G' & \xrightarrow{\psi} & p_2^* G' \end{array}.$$

3.4. **Local (affine) data.** Let
$$\begin{array}{rcl} T & = & SpecA \\ T' & = & SpecA' \end{array}$$
with
$$\begin{array}{c} A''' := A' \otimes_A A' \otimes_A A' \\ \uparrow_{p_{12}} \uparrow_{p_{13}} \uparrow_{p_{23}} \\ A'' := A' \otimes_A A' \\ \uparrow_{p_1} \uparrow_{p_2} \\ A' \\ \uparrow_f \\ A. \end{array} \tag{6}$$
So
$$\begin{array}{rcl} p_1(a') & = & a' \otimes_A 1 \\ p_2(a') & = & 1 \otimes_A a' \end{array}$$

and

$$
\begin{array}{ccc} A' & \stackrel{p_1}{\longrightarrow} & A' \otimes_A A' \\ \uparrow f & & \uparrow p_2 \\ A & \stackrel{f}{\longrightarrow} & A' \end{array} \tag{7}
$$

is a cartesian square, as is

$$
\begin{array}{ccc} A' \otimes_A A' & \stackrel{\pi_{23}}{\longrightarrow} & A' \otimes_A A' \otimes_A A' \\ \uparrow p_1 & & \uparrow p_{12} \\ A' & \stackrel{p_2}{\longrightarrow} & A' \otimes_A A' \end{array} . \tag{8}
$$

Suppose that N' is an A'-module, Then

$$
\begin{array}{rcl} N' \otimes_{p_1} A'' & = & N' \otimes_A A' \\ n' \otimes_{p_1} a' \otimes_A b' & = & n'a' \otimes_A b' \end{array}
$$

and

$$
\begin{array}{rcl} N' \otimes_{p_2} A'' & = & A' \otimes_A N' \\ a' \otimes_A b' \otimes_{p_2} n' & = & a' \otimes_A b'n'. \end{array}
$$

We suppose an A''-module isomorphism

$$
N' \otimes_{p_1} A'' \stackrel{\varphi}{\longrightarrow} N' \otimes_{p_2} A'' \tag{9}
$$

$$
N' \otimes_A A' \stackrel{\varphi}{\longrightarrow} A' \otimes_A N', \tag{10}
$$

that is, if

$$
\varphi(n' \otimes_A 1_{A'}) = \sum a'_i \otimes_A n'_i
$$

then

$$
\begin{array}{rcl} \varphi(n'a' \otimes_A b') & = & \varphi((n' \otimes_A 1_{A'}) \cdot (a' \otimes_A b')) \\ & = & \varphi(n' \otimes_A 1_{A'}) \cdot (a' \otimes_A b') \\ & = & \sum a'_i a' \otimes_A n'_i b'. \end{array}
$$

We further demand that (9) satisfy the cocyle condition

$$
\pi_{23}^* \varphi \circ \pi_{12}^* \varphi = \pi_{13}^* \varphi : N' \otimes_{\pi_1} A''' \to N' \otimes_{\pi_3} A''',
$$

that is, if, for any permutation σ of $\{1, 2, 3\}$, we let

$$
i_\sigma : R'_1 \otimes_A R'_2 \otimes_A R'_3 \to R'_{\sigma 1} \otimes_A R'_{\sigma 2} \otimes_A R'_{\sigma 3}
$$

be the isomorphism which switches factors,

$$
\begin{array}{rcl} \pi_{23}^* \varphi \circ \pi_{12}^* \varphi & = & \pi_{13}^* \varphi : N' \otimes_A A' \otimes_A A' \to A' \otimes_A A' \otimes_A N' \\ (id_{A'} \otimes \varphi) \circ (\varphi \otimes id_{A'}) & = & i_{23} \circ (\varphi \otimes id_{A'}) \circ i_{23} \end{array} \tag{11}
$$

Then we can define a module N as the kernel of the map

$$
\begin{array}{rcl} \varphi \circ (id_{N'} \otimes 1) - (1 \otimes id_{N'}) & : & N' \to A' \otimes_A N' \\ n' & \mapsto & \varphi(n' \otimes_A 1) - (1 \otimes_A n'). \end{array}
$$

Thus

$$
N = \{n' \in N' : \varphi(n' \otimes_A 1) = (1 \otimes_A n')\}
$$

so that, for $n \in N$ we have

$$\begin{aligned} \varphi\left(na' \otimes_A b'\right) &= \varphi\left(\left(n \otimes_A 1\right)\left(a' \otimes_A b'\right)\right) \\ & \left(1 \otimes_A n\right) \cdot \left(a' \otimes_A b'\right) \\ &= \left(a' \otimes_A b'n\right). \end{aligned} \tag{12}$$

so that

$$\varphi\left(na' \otimes_A 1\right) = \left(a' \otimes_A n\right) \tag{13}$$

Via f we have an A-module structure on N together with a natural map

$$h : N \otimes_A A' \to N'. \tag{14}$$

This map is injective since

$$\sum n_i a_i' = 0$$

and

$$\begin{aligned} \varphi\left(\sum n_i a_i' \otimes_A 1_{A'}\right) &= \varphi\left(\sum \left(n_i \otimes_A 1\right)\left(a_i' \otimes_A 1\right)\right) \\ &= \sum \left(1 \otimes_A n_i\right)\left(a_i' \otimes_A 1\right) \\ &= \sum \left(a_i' \otimes_A n_i\right) \end{aligned}$$

implies that

$$\sum n_i \otimes_A a_i' = 0.$$

Lemma 3.4. *Assume that A' is a faithfully flat A-module via f. Then referring to* (14)*, the map*

$$\varphi \circ (h \otimes 1)$$

takes values in

$$1 \otimes N' \subseteq A' \otimes N'$$

and gives an isomorphism of A'-modules

$$N \otimes_A A' \to N' = 1 \otimes N' = N'.$$

Proof. We consider, for any A-module M, the sequence

$$0 \to M \overset{id_M \otimes 1}{\longrightarrow} M \otimes_A A' \overset{(id_M \otimes id_{A'} \otimes 1) - (id_M \otimes 1 \otimes id_{A'})}{\longrightarrow} M \otimes_A A' \otimes_A A' \tag{15}$$

obtained by acting with

$$M \otimes_A$$

applied to

$$0 \to A \to A' \overset{id_{A'} \otimes 1 - 1 \otimes id_{A'}\cdot}{\longrightarrow} A' \otimes_A A'.$$

The sequence (15) is exact by faithful flatness since, applying

$$\otimes_A A',$$

we obtain the sequence

$$\begin{aligned} 0 \quad \to \quad & M \otimes_A A' \overset{id_M \otimes 1 \otimes id_{A'}}{\longrightarrow} M \otimes_A A' \otimes_A A' \\ & \overset{(id_M \otimes id_{A'} \otimes 1 \otimes id_{A'}) - (1_{M'} \otimes 1 \otimes id_{A'} \otimes id_{A'})}{\longrightarrow} M \otimes_A A' \otimes_A A' \otimes_A A' \end{aligned}$$

which is seen to be exact by the fact that $M \otimes_A$ applied to the cartesian square (8) is still a cartesian square. Finally using (15) in the case $M = N'$ as an A-module we obtain

$$0 \to N' \stackrel{id_{N'}\otimes 1}{\longrightarrow} N' \otimes_A A' \stackrel{(id_{N'}\otimes id_{A'}\otimes 1)-(id_{N'}\otimes 1\otimes id_{A'})}{\longrightarrow} N' \otimes_A A' \otimes_A A'. \tag{16}$$

On the other hand the sequence

$$0 \to N \to N' \stackrel{\varphi\circ(id_{N'}\otimes 1)-(1\otimes id_{N'})}{\longrightarrow} A' \otimes_A N'$$

is exact by the definition of N. Also

$$N = \{n' \in N' : \varphi(n' \otimes 1) = 1 \otimes_A n'\}$$

So applying

$$\otimes_A A'$$

to this last sequence and using flatness we obtain the exact sequence

$$0 \to N \otimes_A A' \to N' \otimes_A A' \stackrel{(\varphi\circ(id_{N'}\otimes 1))\otimes id_{A'}-(1\otimes id_{N'})\otimes id_{A'}}{\longrightarrow} A' \otimes_A N' \otimes_A A'. \tag{17}$$

The key to the proof is to compare (17) and (16). To do this, we claim that we have a commutative diagram

$$\begin{array}{ccc} N' \otimes_A A' & \stackrel{(\varphi\circ(id_{N'}\otimes 1))\otimes id_{A'}-1\otimes id_{N'}\otimes id_{A'}}{\longrightarrow} & A' \otimes_A N' \otimes_A A' \\ \downarrow \varphi & & \downarrow id_{A'} \otimes \varphi \\ A' \otimes_A N' & \stackrel{(id_{A'}\otimes 1\otimes id_{N'})-(1\otimes id_{A'}\otimes id_{N'})}{\longrightarrow} & A' \otimes_A A' \otimes_A N' \end{array}$$

with exact rows and vertical isomorphisms. It suffices to check that

$$\begin{array}{ccc} N' \otimes_A A' & \stackrel{(\varphi\circ(id_{N'}\otimes 1))\otimes id_{A'}}{\longrightarrow} & A' \otimes_A N' \otimes_A A' \\ \downarrow \varphi & & \downarrow id_{A'} \otimes \varphi \\ A' \otimes_A N' & \stackrel{(id_{A'}\otimes 1\otimes id_{N'})}{\longrightarrow} & A' \otimes_A A' \otimes_A N' \end{array} \tag{18}$$

is commutative. Recall that, by (11)

$$(id_{A'} \otimes \varphi) \circ (\varphi \otimes id_{A'}) = i_{23} \circ (\varphi \otimes id_{A'}) \circ i_{23}.$$

Now apply the two formulas for this last map to

$$N' \otimes_A 1 \otimes_A A'.$$

Finally we use (18) to obtain a commutative diagram

(19)

$$\begin{array}{ccccc} N \otimes_A A' & \to & N' \otimes_A A' & \stackrel{\substack{(\varphi\circ(id_{N'}\otimes 1))\otimes id_{A'} \\ -(1\otimes id_{N'})\otimes id_{A'}}}{\longrightarrow} & A' \otimes_A N' \otimes_A A' \\ & & \downarrow \varphi & & \downarrow id_{A'} \otimes \varphi \\ N' & \stackrel{id_{N'}\otimes 1}{\longrightarrow} & A' \otimes_A N' & \stackrel{\substack{(id_{A'}\otimes 1\otimes id_{N'}) \\ -(1\otimes id_{A'}\otimes id_{N'})}}{\longrightarrow} & A' \otimes_A A' \otimes_A N' \end{array}$$

But by (13) we have

$$\varphi(na' \otimes_A 1) = (a' \otimes_A n)$$

which implies that the image of $N \otimes_A A'$ in $A' \otimes_A N'$ under the natural inclusion

$$\begin{array}{ccc} N \otimes_A A' & \to & A' \otimes_A N' \\ n \otimes a' & \mapsto & a' \otimes n \end{array}$$

is the kernel of the map

$$A' \otimes_A N' \overset{(id_{A'} \otimes 1 \otimes id_{N'}) - (1 \otimes id_{A'} \otimes id_{N'})}{\longrightarrow} A' \otimes_A A' \otimes_A N'.$$

But by (16) that kernel is exactly $1 \otimes N' = N'$. To show that the induced isomorphism

$$N \otimes_A A' \to N'$$

is the standard one h defined above, we write

$$\begin{aligned} \varphi(n \otimes_A a') &= \varphi((n \otimes 1) \cdot (1 \otimes a')) = (\varphi(n \otimes 1)) \cdot (1 \otimes a') \\ &= (1 \otimes n) \cdot (1 \otimes a') = 1 \otimes na'. \end{aligned}$$

□

Once we have N constructed from descent data, notice that, writing N' as $N \otimes_A A'$, then

(20) $$\begin{array}{c} \varphi : N \otimes_A A' \otimes_A A' \to A' \otimes_A N \otimes_A A'. \\ n \otimes a' \otimes b' \mapsto a' \otimes n \otimes b' \end{array}$$

3.5. **Local data for morphisms.** We continue to assume that

$$\otimes_A A'$$

is a faithfully flat functor. Given descent data

$$(N', \varphi), (M', \psi)$$

then the sequence

$$0 \to Hom(M, N) \overset{\otimes_A A'}{\longrightarrow} Hom(M', N') \underset{\otimes_{p_2} A'}{\overset{\otimes_{p_1} A'}{\longrightarrow}} Hom(M', N')$$

is exact. To see this, first recall that

$$\begin{array}{c} Hom(M \otimes_A A', N \otimes_A A') \overset{\otimes_{p_1} A'}{\longrightarrow} Hom(M \otimes_A A' \otimes_A A', N \otimes_A A' \otimes_A A') \\ ((m \otimes 1) \mapsto (\sum n_i \otimes a_i')) \mapsto ((m \otimes 1 \otimes 1) \mapsto (\sum n_i \otimes a_i' \otimes 1)) \end{array}$$

so that the above sequence is just

$$\begin{array}{l} Hom(M, N) \overset{\otimes_A A'}{\longrightarrow} \\ \qquad Hom(M \otimes_A A', N \otimes_A A') \overset{id_{A'} \otimes 1 - 1 \otimes id_{A'}}{\longrightarrow} Hom(M \otimes_A A' \otimes_A A', N \otimes_A A' \otimes_A A'). \\ \qquad\qquad ((m \otimes 1) \mapsto (\sum n_i \otimes a_i')) \mapsto ((m \otimes 1 \otimes 1) \mapsto (\sum n_i \otimes (a_i' \otimes 1 - 1 \otimes a_i'))) \end{array}$$

So exactness comes from the fact that, if

$$\sum n_i \otimes (a_i' \otimes 1 - 1 \otimes a_i') = 0$$

then

$$\left(\sum n_i \otimes a_i'\right) \in \ker((id_{N'} \otimes id_{A'} \otimes 1) - (id_{N'} \otimes 1 \otimes id_{A'}))$$

so that

$$\left(\sum n_i \otimes a_i'\right) \in N$$

by (20).

3.6. **Globalization.** We globalize this situation by considering a faithfully flat quasi-compact morphism of schemes

$$f' : T' \to T$$

and so have a cartesian diagram

$$\begin{array}{ccc} T'' := T' \times_T T' & \xrightarrow{p_2} & T' \\ \downarrow p_1 & \searrow^{\overline{f}} & \downarrow f \\ T' & \xrightarrow{f} & T \end{array} .$$

Then (6) globalizes to the left exact sequence

$$0 \to \mathcal{O}_T \xrightarrow{f^{\#}} f_* \mathcal{O}_{T'} \xrightarrow{f_*\left(p_1^{\#} - p_2^{\#}\right)} \overline{f}_* \mathcal{O}_{T''}.$$

Given an affine cover

$$\{T_i\}$$

of T and affine cover

$$\{T'_{ij}\}$$

of each T_i, then by descent, a quasi-coherent sheaf $\mathfrak{F}$ on T is a datum

$$(\mathfrak{F}', \varphi : p_1^* \mathfrak{F}' \to p_2^* \mathfrak{F}') .$$

4. Descent and stacks

We return to our situation of a category fibered in groupoids

$$F \to \mathfrak{S}$$

over the category of schemes. Given a faithfully flat quasi-compact morphism of schemes

$$f' : T' \to T$$

we have the globalization

$$\begin{array}{c} T''' := T' \times_T T' \times_T T' \\ \downarrow_{\pi_{12}} \downarrow_{\pi_{13}} \downarrow_{\pi_{23}} \\ T'' := T' \times_T T' \\ \downarrow_{p_1} \downarrow_{p_2} \\ T' \\ \downarrow_f \\ T. \end{array} \tag{21}$$

of (6). We define a category

$$F^d_{T' \to T}$$

by

$$Obj\left(F^d_{T'\to T}\right) = \left\{\varphi : p_1^* u \xrightarrow{\cong} p_2^* u : u \in F_{T'}, \pi_{23}\varphi \circ \pi_{12}\varphi = \pi_{13}\varphi\right\}$$

$$Mor\left(F^d_{T'\to T}\right) = \left\{ h : (u,\varphi) \to (v,\psi) : \begin{array}{ccc} p_1^* u & \xrightarrow{p_1^* h} & p_1^* v \\ \downarrow \varphi & comm. & \downarrow \psi \\ p_2^* u & \xrightarrow{p_2^* h} & p_2^* v \end{array} \right\}.$$

and a "descent" or "derivative" functor

$$\begin{array}{rcl} df & : & F_T \to F^d_{T'\to T}. \\ t & \mapsto & \left(f^* t, p_1^* f^* t \xrightarrow{\cong} p_2^* f^* t\right) \end{array}$$

Notice that this functor is well-defined since

$$f \circ p_1 = f \circ p_2$$

and so, by the property 2 in the definition of categories fibered in groupoids, the isomorphism

$$\begin{array}{ccc} p_1^* f^* t & & \\ & \searrow^{(f\circ p_1)^*} & \\ \downarrow \cong & & t \\ & \nearrow_{(f\circ p_2)^*} & \\ p_2^* f^* t & & \end{array}$$

$$p_1^* f^* t \xrightarrow{\cong} p_2^* f^* t$$

is canonically determined by the diagrams

$$\begin{array}{ccc} T'' & & \\ & \searrow^{f\circ p_1} & \\ \downarrow id. & & T \\ & \nearrow_{f\circ p_2} & \\ T'' & & \end{array}$$

in $\mathfrak{S}$, and liftings $p_1^* f^* t \in l\,(t, f \circ p_1)$ and $p_2^* f^* t \in l\,(t, f \circ p_2)$.

Thus we have a redefinition of stack in the etale topology:

Definition 4.1. A stack in the etale topology is a category F fibered in groupoids such that, for single map etale surjective faithfully flat quasi-compact covers

$$f : T' \to T$$

the functor

$$df : F_T \to F^d_{T'\to T}$$

is an equivalence of categories and

$$F_{\coprod V_\alpha} = \prod_\alpha F_{V_\alpha}.$$

In short, this definition is simply that a category fibered in groupoids over the category of schemes is a fibered category in which "descent rules." Notice that the "single cover" restriction is no restriction at all since any cover can be made into a single cover by taking the disjoint union of the domains of the covering as a new single domain. Essentially our local computations above, together with an essentially infinite number of commutative diagram checks gives:

Lemma 4.1. *The fibered category*

$$Qcoh \to \mathfrak{S}$$

is a category whose objects over $S \in \mathfrak{S}$ are the set of quasicoherent sheaves on the scheme S. An arrow from a sheaf G on T to a sheaf F on S is a morphism

$$f : T \to S$$

and an isomorphism

$$G \cong f^* F.$$

This fibered category is a stack.

Proposition 4.2.

$$\underline{X}$$

is a stack.

Proof. We need to check that

$$\underline{X}_T \to \underline{X}^d_{T' \to T}$$

is an equivalence of categories. The question is whether, for

$$f : T' \to T$$

the morphisms

$$u : T \to X$$

are equivalent to descent data

$$u' : T' \to X$$

with

$$u' \circ p_1 = u' \circ p_2. \tag{22}$$

Now take affine covers

$$\begin{aligned} T &= \bigcup\nolimits_i T_i \\ T' &= \bigcup\nolimits_{ij} T'_{ij} \end{aligned}$$

where

$$f\left(T'_{ij}\right) \subseteq T_i.$$

Then

$$0 \to k\left[T_i\right] \to \bigoplus\nolimits_j k\left[T'_{ij}\right] \to \left(\bigoplus\nolimits_j k\left[T'_{ij}\right]\right) \otimes_{k[T_i]} \left(\bigoplus\nolimits_j k\left[T'_{ij}\right]\right) \tag{23}$$

is exact by faithful flatness since the sequence obtained by tensoring it with

$$\otimes_{k[T_i]} \left(\bigoplus\nolimits_j k\left[T'_{ij}\right]\right)$$

is exact. We can localize (23) as the exact sequence

$$0 \to \mathcal{O}_{T_i} \to \bigoplus\nolimits_j f_* \mathcal{O}_{T'_{ij}} \to \left(\bigoplus\nolimits_j f_* \mathcal{O}_{T'_{ij}}\right) \otimes_{\mathcal{O}_{T_i}} \left(\bigoplus\nolimits_j f_* \mathcal{O}_{T'_{ij}}\right)$$

Let

$$u'_i = u'|_{f^{-1}(T_i)}.$$

Now by (22) we have a morphism

$$f_*\left(u_i'\right)^{-1}\mathcal{O}_X \to \ker\left(\bigoplus_j f_*\mathcal{O}_{T'_{ij}} \to \left(\bigoplus_j f_*\mathcal{O}_{T'_{ij}}\right)\otimes_{\mathcal{O}_{T_i}}\left(\bigoplus_j f_*\mathcal{O}_{T'_{ij}}\right)\right)$$

which defines a Zariski-continuous map

$$u_i : T_i \to X$$

and so a morphism

$$\mathcal{O}_X \to \left(u_i\right)_*\left(\mathcal{O}_{T_i}\right).$$

Use the uniqueness of the construction to paste the locally defined maps together. □

Proposition 4.3. *If G is an affine algebraic group,*

$$BG$$

is a stack.

Proof. Given descent data

$$E' \to T'$$

with

$$\varphi : E'' \to E''$$

we want

$$E \to T.$$

Recall that local triviality means

$$E' \cong G \times T'$$

so that

$$\varphi : G \times T'' \to G \times T''$$

is an isomorphism of G-bundles, that is, a map

$$\tau : T'' \to G.$$

The cocycle condition

$$G \times T''' \to G \times T'''$$

becomes

$$\tau\left(t,v\right) = \tau\left(u,v\right)\cdot\tau\left(t,u\right).$$

Thus we can define and equivalence relation

$$\left(g_1, t_1'\right) \sim \left(g_2, t_2'\right)$$

if and only if

$$\left(t_1, t_2\right) \in T' \times_T T' = T''$$

and

$$g_2 = \tau\left(t_1, t_2\right)\cdot g_1.$$

The exactness of (23) then gives

$$\mathcal{O}_T \to \mathcal{O}_G \otimes \mathcal{O}_{T'}$$

so that the compostions with the maps

$$p_i^{\#} : \mathcal{O}_G \otimes \mathcal{O}_{T''} \to \mathcal{O}_G \otimes \mathcal{O}_{T''}$$

correspond via τ. We then apply a G-equivariant version of (4.1) (see Theorem 3.3). □

5. Algebraic spaces

Definition 5.1. An *equivalence relation* in the category of schemes over S is a pair of schemes over S

$$(R, U)$$

with two morphisms of schemes

$$R \overset{s}{\underset{t}{\rightrightarrows}} U,$$

or equivalently a single morphism

$$R \to U \times_S U \tag{24}$$

locally of finite type (that is, the inverse image of each scheme in some affine cover is quasi-compact, i.e., covered by a finite number of affines) such that

i) (24) is *categorically injective*, that is, if the compostions

$$T \rightrightarrows R \to U \times_S U$$

coincide then two maps

$$T \rightrightarrows R$$

coincide,

ii)

$$R(T) = Hom_S(T, R) \subseteq U(T) \times U(T)$$

is an equivalence relation.

Condition ii) just above can be reexpressed in terms of the standard conditions for an equivalence relation as follows:

iia) The diagonal map

$$U \to U \times_S U$$

lifts to R (and so in particular the two maps

$$s, t : R \rightrightarrows U$$

are surjective),

iib) the involution

$$\begin{array}{ccc} U \times_S U & \to & U \times_S U \\ (x, y) & \mapsto & (y, x) \end{array}$$

lifts to an involution on R;

iic) since the diagram

$$\begin{array}{ccc} U \times_S U \times_S U & \xrightarrow{\pi_{23}} & U \times_S U \\ \downarrow \pi_{12} & & \downarrow \pi_1 \\ U \times_S U & \xrightarrow{\pi_2} & U \end{array}$$

is cartesian, it receives a morphism from the cartesian diagram

$$\begin{array}{ccc} R\times_U R & \to & R \\ \downarrow & & \downarrow^{t} \\ R & \xrightarrow{s} & U \end{array}$$

and the requirement is that the diagram

$$\begin{array}{ccc} R\times_U R & & R \\ \downarrow & & \downarrow \\ U\times_S U\times_S U & \to & U\times_S U \\ (x,y,z) & \mapsto & (x,z) \end{array}$$

be made commutative by a lift

$$R\times_U R\to R.$$

Notice that all lifts in the above definition are unique because of categorical injectivity.

Definition 5.2. An equivalence relation

$$R\overset{(s,t)}{\longrightarrow}U\times_S U$$

is etale if at least one of the (locally finite type) maps s or t is etale.

Given an etale equivalence relation

$$R\rightrightarrows U$$

one defines a *quotient presheaf* as a functor

$$P:\underline{S}^{\circ}\to Sets$$

with

$$P(T)=\frac{U(T)}{R(T)}$$

that is, equivalence classes of map

$$T\to U.$$

This presheaf is *separated*, that is, for an etale cover

$$T_\alpha\to T$$

the induced map

$$P(T)\to\prod P(T_\alpha)=P\left(\coprod T_\alpha\right)$$

is injective

Definition 5.3. An *algebraic space* U/R is the sheafification in the etale topology of the presheaf P derived from

$$R\to U\times_S U \tag{25}$$

as above. If (25) is a closed immersion, we call the algebraic space *separated.* If (25) is simply and immersion, we call the algebraic space *locally separated.*

So an algebraic space is a functor

$$\begin{array}{rcl} U/R & : & \underline{S}^{\circ} \to sets \\ T & \mapsto & (U/R)\,(T) \end{array}$$

An element of

$$(U/R)\,(T)$$

is an equivalence class of pairs

$$\left(t' : T' \overset{etale,\ surj.}{\longrightarrow} T,\ \tau' \in P\,(T')\right)$$

where the equivalance relation is given by

$$(t', \tau') \sim (t'', \tau'')$$

if the map

$$T' \times_T T'' \overset{(\tau', \tau'')}{\longrightarrow} U \times_S U$$

factors through R.

Notice that we have a morphism (natural transformation)

$$\begin{array}{rcl} \underline{U} & \Rightarrow & U/R \\ (T \to U) & \mapsto & (U/R)\,(T) \end{array} \tag{26}$$

Proposition 5.1. *The map*

$$\underline{R} \to \underline{U} \times_{U/R} \underline{U}$$

induced by

$$\begin{array}{ccc} \underline{R} & \to & \underline{U} \\ \downarrow & & \downarrow \\ \underline{U} & \to & U/R \end{array}$$

is an isomorphism.

Proof. The natural map

$$\frac{U\,(T)}{R\,(T)} \to (U/R)\,(T)$$

is an injection, so

$$R\,(T) = U\,(T) \times_{(U/R)(T)} U\,(T)\,.$$

□

Lemma 5.2. *The natural transformation* (26) *is represented by (etale and surjective) schemes, that is, if*

$$\underline{X} \to U/R$$

is a morphism, then

$$\underline{U} \times_{U/R} \underline{X} \leftrightarrow \underline{Y}$$

where Y is a scheme and

$$\underline{Y} \to \underline{X}$$

induced by

$$Y \to X$$

etale and surjective.

Proof. First assume we have a factorization

$$\underline{X} \to \underline{U} \to U/R.$$

The cartesian diagram

$$\begin{array}{ccccc} & & & & \underline{X} \\ & & & & \downarrow \\ \underline{R} & = & \underline{U} \times_{U/R} \underline{U} & \to & \underline{U} \\ & & \downarrow & & \downarrow \\ & & \underline{U} & \to & U/R \end{array}$$

shows that

$$\underline{U} \times_{U/R} \underline{X} = \underline{R \times_U X}$$

and the projection

$$R \times_U X \to X$$

is etale and surjective since

$$R \to U$$

is.

In general there exist an X' and a commutative diagram

$$\begin{array}{ccccc} \underline{X'} & \to & \underline{U} & \leftarrow & \underline{R} \\ \downarrow & & \downarrow & & \downarrow \\ \underline{X} & \to & U/R & \leftarrow & \underline{U} \end{array}$$

with

$$X' \to X$$

etale and surjective. So, by the first step, we get

$$\underline{U} \times_{U/R} \underline{X'} = \underline{R \times_U X'}.$$

One then constructs the scheme representation of

$$\underline{U} \times_{U/R} \underline{X}$$

by descent for etale maps using descent data

$$\begin{array}{ccc} \underline{U} \times_{U/R} \left(\underline{X' \times_X X'}\right) & & \underline{X' \times_X X'} \\ \downdownarrows & & \downdownarrows \\ \underline{U} \times_{U/R} \underline{X'} & \to & \underline{X'} \end{array} .$$

The (rather difficult) descent theorem for etale maps then produces a scheme

$$U \times_{U/R} X$$

which is etale and surjective over X. ☐

Example 5.1. Let U be given by

$$xy = 0$$

in the affine plane and let R be given by

$$(x, 0) = (0, x)\,,\ x \neq 0.$$

Then U/R is the affine line with an “extra” formal neighborhood of 0 in $\mathbb{A}^1$ attached at 0. It is a locally separated algebraic space but is not separated since one component of R has non-closed image in U. Notice that the morphism

$$U/R \to \mathbb{A}^1$$

is ramified but

$$\underline{U} \to U/R$$

is not.

Example 5.2. For

$$S = Spec\,(\mathbb{C})$$

we have that

$$F\,(Spec\,(\mathbb{C})) = \frac{U\,(Spec\,(\mathbb{C}))}{R\,(Spec\,(\mathbb{C}))}$$

since every etale surjective map

$$T' \to Spec\,(\mathbb{C})$$

has a section.

Now we can make the following alternative definition of algebraic space.

Definition 5.4. An algebraic space is a sheaf in the etale topology, that is, a functor

$$F : \underline{S}^{\circ} \to Sets,$$

such that there exists a scheme V and a morphism

$$\underline{V} \to F$$

which is represented by an (etale and surjective) scheme and is such that

$$\underline{V} \times_F \underline{V} \to \underline{V \times_S V}$$

is quasi-finite (that is, finite fibers).

We have already seen that the original definition implies the alternative one. In the other direction the representability condition imples that

$$R = V \times_F V$$

is a scheme of finite type over S and then one checks that R is an etale equivalence relation. So we have an algebraic space V/R. To see that

$$F \leftrightarrow V/R$$

look at the quotient presheaf

$$P = \frac{V\,(T)}{R\,(T)}$$

giving, by definition, an injective morphism of presheaves

$$P \to F.$$

Then one only needs to check that every section comes locally from P, that is, for some etale surjective

$$T' \to T$$

we have that our element of $F(T')$ lifts to $V(T')$. But we can achieve this by letting

$$T' = V \times_F T.$$

Thus an algebraic space is a sheaf which is locally a scheme in the etale topology, and so every property of schemes which is local in the etale topology translates to a property of algebraic spaces.

Lemma 5.3. *If F is an algebraic space, the diagonal*

$$F \to F \times_S F$$

is represented by a scheme.

Proof. Let

$$\underline{X'} = \underline{X} \times_F \underline{U}$$

and note that

$$X' \to X$$

is quasi-finite. Consider the diagram

$$\begin{array}{cccccccc}
 & & & fp_3 & & \to & & \underline{X' \times_X X'} \\
 & & \swarrow\swarrow & & & & \swarrow\swarrow & \\
 & fp_2 & & \to & & \underline{X'} & & \\
\swarrow & | & & & \swarrow & & & \\
fp_1 & \to & & \underline{X} & & \downarrow & & \\
 & \downarrow & & & & & & \\
\downarrow & R & - & \downarrow & \to & \underline{U \times_S U} & & \\
\swarrow & & & & \swarrow & & & \\
F & \to & & F \times_S F & & & &
\end{array}$$

where fp denotes fibered product. Since fp_2 and fp_3 are schemes, so is fp_1 by Grothendieck's descent theorem for quasi-finite maps. □

Corollary 5.4. *If F is an algebraic space and X and Y are schemes for which there are morphisms*

$$\begin{array}{ccc} & & \underline{X} \\ & & \downarrow \\ \underline{Y} & \to & F \end{array}$$

then

$$\underline{X} \times_F \underline{Y}$$

is a scheme.

Proof. We have a cartesian square

$$\begin{array}{ccc} \underline{X} \times_F \underline{Y} & \to & \underline{X \times_S Y} \\ \downarrow & & \downarrow \\ F & \to & F \times_S F \end{array}.$$

□

Proposition 5.5. *Let F, F', G be algebraic spaces over S. Then*

$$F \times_G F'$$

is an algebraic space.

Proof. By the above we have

$$\begin{array}{ccc} \underline{U} & \to & F \\ \underline{U'} & \to & F' \end{array}$$

representable by schemes, etale and surjective. Thus, by above,

$$\underline{U} \times_G \underline{U'} \to F \times_G F'$$

is representable by schemes. But $\underline{U} \times_G \underline{U'}$ is a scheme V and its diagonal

$$\underline{R} \times_G \underline{R'}$$

is of finite type over S and so becomes the R for the algebraic space

$$R \to V \times_S V.$$

We now clarify the notion of separability which, in the above definition of algebraic space, seemed to depend on the choice of U. □

Definition 5.5. An algebraic space F is *locally separated* if the diagonal map

$$F \to F \times_S F$$

is an immersion and *separated* if it is a closed immersion.

To justify this definition in light of our earlier use of the terms, we need:

Proposition 5.6. *If*

$$\underline{U} \to F$$

is etale and surjective, then F is locally separated (resp. separated) if and only if

$$\underline{R} = \underline{U} \times_F \underline{U} \to \underline{U \times_S U}$$

is an immersion (resp. closed immersion).

Proof. (Closed) immersion is a local property in the etale topology (on the domain) and stable under base change. So we can appeal to the following proposition. □

Proposition 5.7. *Let*

$$F \to G$$

be a morphism of algebraic spaces represented by schemes. Let P be a property of morphisms of schemes which is local in the etale topology on the domain and (faithfully) stable under etale, surjective base change, and if

$$\underline{V} \to G$$

is etale and surjective. Then the morphism of schemes

$$F \times_G \underline{V} \to \underline{V}$$

has property P if and only if, for all etale surjective morphisms

$$\underline{X} \to G,$$

X a scheme,

$$F \times_G \underline{X} \to \underline{X}$$

has property P.

Proof. Consider the diagram

$$\begin{array}{ccccc}
 & & F \times_G \underline{X} \times_G \underline{V} & \to & \underline{X} \times_G \underline{V} \\
 & \swarrow & & \swarrow & \downarrow \\
F \times_G \underline{X} & \to & \underline{X} & & \underline{V} \\
\downarrow & & \downarrow & \swarrow & \\
F & \to & G & &
\end{array}$$

in which all squares are cartesian. Given that

$$F \times_G \underline{V} \to \underline{V}$$

has P, then

$$F \times_G \underline{V} \times_G \underline{X} \to \underline{V} \times_G \underline{X}$$

has P so

$$F \times_G \underline{X} \to \underline{X}$$

does too because

$$\underline{X} \times_G \underline{V} \to \underline{X}$$

is etale and surjective. □

Finally there is a rather deep result about "representing" locally separated algebraic spaces as analytic varieties in the case $S = Spec\,(\mathbb{C})$:

Theorem 5.8. *If $R \rightrightarrows U$ is a locally separated algebraic space, then via the quotient topology on*

$$U\,(Spec\,(\mathbb{C})) \to \frac{U\,(Spec\,(\mathbb{C}))}{R\,(Spec\,(\mathbb{C}))} = F\,(Spec\,(\mathbb{C}))\,,$$

there exists a unique analytic structure on $F\,(Spec\,(\mathbb{C}))$ making

$$U\,(Spec\,(\mathbb{C})) \to F\,(Spec\,(\mathbb{C}))$$

a local analytic isomorphism.

Proof. Based on the fact that, given that

$$R \to U \times_S U$$

is an immersion, then

$$R\,(Spec\,(\mathbb{C})) \to U\,(Spec\,(\mathbb{C})) \times U\,(Spec\,(\mathbb{C}))$$

is injective and has the property that, for any point $x \in U\,(Spec\,(\mathbb{C}))$, there is a neighborhood A of x such that

$$(A \times A) \cap R\,(Spec\,(\mathbb{C})) \subseteq \Delta$$

where Δ is the diagonal of $A \times A$. □

5.1. Global properties of algebraic spaces and morphisms.

Definition 5.6. An algebraic space F is of *finite type* over S if there exists an etale surjective morphism

$$\underline{U} \to F$$

with U a scheme of finite type over S.

Definition 5.7. The *image* of an algebraic space F/S is the image of the morphism of schemes given by the composition

$$\underline{U} \to F \to \underline{S}$$

where $\underline{U} \to F$ is etale, surjective.

Definition 5.8. An algebraic space F is *proper* over S if it is of finite type, separated, and universally closed, that is, the image of all base extensions

$$F' \to F$$

is closed in S.

Proposition 5.9. *An (open, closed, locally closed) subspace G of an algebraic space F is an algebraic space G with a morphism*

$$G \to F$$

represented by (open, closed, locally closed) embeddings, that is, every fibered product

$$\begin{array}{ccc} G \times_F \underline{X} & \xrightarrow{\alpha} & \underline{X} \\ \downarrow & & \downarrow \\ G & \to & F \end{array}$$

has the property that α is an (open, closed, locally closed) embedding of schemes.

Suppose now that

$$G \hookrightarrow F$$

is an open subspace, then for the open subspace

$$\underline{V} = \underline{U} \times_F G$$

we have the diagram

$$\begin{array}{ccc} \underline{R} & = & \underline{U} \times_F \underline{U} \\ & & {}^{t}\downdownarrows^{s} \\ \underline{V} & \to & \underline{U} \\ \downarrow & & \downarrow \\ G & \to & F \end{array} .$$

Then

$$s^{-1}(V) = t^{-1}(V) = R \times_U V.$$

Definition 5.9. An open subscheme $V \subseteq U$ is called *invariant* if

$$s^{-1}(V) = t^{-1}(V).$$

Exercise 5.1. V is invariant if and only if

$$V = s\left(t^{-1}(V)\right).$$

Exercise 5.2. If V is an open subscheme of U then

$$s\left(t^{-1}(V)\right)$$

is the smallest invariant open subscheme of U containing V.

Exercise 5.3. If V is an invariant open subset of U then

$$R_V = s^{-1}(V) = t^{-1}(V) \subseteq R$$

is an equivalence relation on V, then

$$V/R_V \to F$$

is an open embedding and

$$\underline{V} = V/R_V \times_F \underline{U}.$$

If, for sone algebraic space G we have that

$$\underline{V} = G \times_F \underline{U},$$

then

$$G = V/R_V.$$

A foundational result of Grothendieck is:

Theorem 5.10. *If $R \rightrightarrows U$ is an equivalence relation such that R and U are affine and either map (and hence both maps) $R \to U$ is finite and flat, then*

$$U/R$$

is an affine scheme.

Corollary 5.11. *If F is an algebraic space of finite type over a noetherian scheme S, then there exists an open dense subspace*

$$G \hookrightarrow F$$

which is a scheme.

Proof. We have

$$\underline{U} \to F$$

etale and surjective with U/S of finite type. Then

$$R = U \times_F U \to U$$

is etale and of finite type and so quasi-finite. Now take the largest V' open in U so that

$$t^{-1}(V') \to V'$$

is finite. Then V is invariant so that

$$V'/V' \times_F V$$

is an open dense subspace of F.

So by restricting to an affine dense V'' in V' and setting

$$V = s\left(t^{-1}(V'')\right)$$

we obtain by the Theorem 5.10 an affine open dense subscheme

$$G = V/V \times_F V.$$

∎

The above local and global properties of morphisms of algebraic spaces imply that one can glue algebraic spaces together in the Zariski topology, that is, if

$$F \rightrightarrows G$$

is an equivalence relation in the category of algebraic spaces, then

$$G/F$$

is an algebraic space.

5.2. **Hironaka's example.** Let X be a projective threefold with a fixpoint-free involution

$$\sigma : X \to X$$

and $C \subset X$ is a smooth irreducible curve such that C and $\sigma(C) = C'$ meet transversely at two points p and p'. For example X might be $Pic^2(C_0)$ for a hyperelliptic curve C_0 of genus 3,

$$\sigma = y - x$$

where x and y are distinct Weierstrass points of C_0 and

$$\begin{aligned} C &= x + C_0 \\ C' &= y + C_0. \end{aligned}$$

Let B be the variety obtained from $X - \{p'\}$ by first blowing up C and then blowing up the proper transform of C'. Let

$$q \in B$$

denote the point at which the proper transform E of the fiber of the first blow-up over p meets the exceptional locus of the second blow-up. Let $B \sqcup B'$ denote the disjoint union of two copies of B and define the (proper, smooth, non-projective) algebraic variety

$$\tilde{X} = \frac{B \sqcup B'}{(b \in B) \sim (\sigma(b) \in B')}.$$

Then σ lifts to a fixpoint-free involution $\tilde{\sigma}$ on $\tilde{X}$. Let

$$\begin{aligned} q' &= \tilde{\sigma}(q) \\ E' &= \tilde{\sigma}(E). \end{aligned}$$

There is no affine subscheme of $\tilde{X}$ which contains both q and q' since its complement would have to be a divisor which meets the curve

$$E + E'$$

with positive intersection number. But $E + E'$ is numerically the zero cycle.

Now define the closed imbedding

$$R = \tilde{X} \sqcup \tilde{X} \hookrightarrow \tilde{X} \times \tilde{X} = U \times U$$

where the map on the first component is the diagonal map and the second map is given by $(identity, \tilde{\sigma})$. This gives

$$F = U/R$$

the structure of a separated, proper, smooth algebraic space with etale surjective morphism

$$\pi : \underline{\tilde{X}} \to F.$$

There exists a closed subspace

$$G \hookrightarrow F$$

with

$$\begin{array}{ccc} \{q, q'\} & \to & \underline{\tilde{X}} \\ \downarrow & & \downarrow \\ G & \to & F \end{array} .$$

It is instructive to notice that no open neighborhood of the point G is a scheme since, by shrinking, we could assume the scheme to be affine so that its inverse image would be an affine subscheme of $\underline{\tilde{X}}$ containing q and q'.

6. Dictionary

Definition 6.1. A *prestack* is a category fibered in groupoids over the category of schemes such that, for all $x, x' \in \mathfrak{C}_X$, the functor

$$I := Iso_X(x, x') : \underline{X} \to Sets$$

is a sheaf with respect to a chosen Grothendieck topology.

A *stack* is a prestack for which "descent data is effective," which roughly means that, for each scheme X, the "functor"

$$\mathfrak{C} : \underline{X} \to Sets$$
$$(F : S \to X) \mapsto \mathfrak{C}_S$$

is a "sheaf" for the chosen Grothendieck topology.

Definition 6.2. A stack $\mathfrak{C}$ is representable if there is a fully faithful, essentially surjective morphism

$$\underline{U/R} \cong \mathfrak{C}$$

for some algebraic space U/R. A morphism

$$\mathfrak{C} \to \mathfrak{C}'$$

of stacks is representable if, for every morphism of stacks

$$\underline{U/R} \to \mathfrak{C},$$

the fibered product

$$\underline{U/R} \times_{\mathfrak{C}'} \mathfrak{C}$$

is representable.

Definition 6.3. A stack $\mathfrak{C}$ is *algebraic* if:

i) the diagonal morphism

$$\mathfrak{C} \to \mathfrak{C} \times_{\mathfrak{S}} \mathfrak{C}$$

is representable by a quasi-compact (separated) algebraic space;

ii) there exists a cover of $\mathfrak{C}$ by a scheme, that is a scheme U and a representable morphism

$$\underline{U} \to \mathfrak{C}$$

such that, for every scheme U' with

$$\underline{U'} \to \mathfrak{C},$$

the fibered product

$$\underline{U} \times_{\mathfrak{C}} \underline{U'}$$

is a cover of

$$\underline{U'}$$

in the chosen Grothendieck topology.

Corollary 6.1. *Every algebraic stack $\mathfrak{C}$ is of the form*

$$\underline{U} \times_{\mathfrak{C}} \underline{U} \rightrightarrows \underline{U}$$

for some scheme U. That is

$$\underline{R} \cong \underline{U} \times_{\mathfrak{C}} \underline{U}.$$

Definition 6.4. An algebraic stack for the Grothedieck topology of etale maps is called a *Deligne-Mumford stack.*

An algebraic stack for the Grothedieck topology of smooth maps is called an *Artin stack.*

In the context of the last definition, it is useful to recall that all our schemes are locally of finite type and to recall the **Grothendieck criterion for smoothness or etaleness of a map**:

A morphism

$$f : X \to Y$$

of schemes is *smooth* if and only if, given any local Artinian ring A and quotient ring

$$B := \frac{A}{I}$$

and commutative diagram

$$\begin{array}{ccc} SpecB & \to & X \\ \downarrow & & \downarrow^{f} \\ SpecA & \to & Y \end{array},$$

there is a lifting

$$SpecA \to X$$

making the full diagram commutative. The morphism f is *etale* if, for any diagram as above, the lifting exists and is unique. The morphism f is *unramified* if, for any diagram, there is at most one lifting.

Thus, for example, if $\mathfrak{C}$ is an algebraic stack, a morphism

$$\underline{X} \to \mathfrak{C}$$

is *smooth* if, for every commutative diagram

$$\begin{array}{ccc} \underline{SpecB} & \xrightarrow{\psi^*} & \underline{X} \\ \downarrow & & \downarrow^{f} \\ \underline{SpecA} & \xrightarrow{\psi} & \mathfrak{C} \end{array}$$

there exists a lifting

$$\tau : \underline{SpecA} \to \underline{X}$$

such that

$$\tau|_{\underline{SpecB}} \cong \psi^*.$$

We say that the family $\underline{X}$ is *versal* at

$$\psi^* \left(\underline{Spec\frac{B}{\mathfrak{m}}} \right).$$

Notice that, if a functor

$$F : Schemes^\circ \to Sets$$

is to be a sheaf, there must be a well-defined set

$$F(S)$$

so that, if

$$f : S \to S$$

is an isomorphism, the induced isomorphism

$$F(f) : F(S) \to F(S)$$

is well-defined. Thus, for example, if

$$F = Hom(\ , X)$$

and, for all

$$s \in Hom(S, X)$$

we have

$$s = s \circ f,$$

then it had better be that

$$F(f) = 1_{F(S)}$$

If our schemes are over a field k. then the set

$$F(k)$$

is the set of "points" of X.

Definition 6.5. A *separated algebraic space* is a covariant functor

$$\begin{array}{rcl} F & : & \mathfrak{S}^\circ \to Sets \\ S & \mapsto & Hom(S, X) \end{array}$$

such that

(0) F is a sheaf for the etale topology;

(1)

$$\Delta : F \to F \times_{\mathfrak{S}} F$$

is representable by a scheme;

(2) F has an affine cover in the etale topology.

In short, an algebraic space is a functor X, such that there exists a representable etale morphism from a scheme onto X. The scheme is not part of the data, the functor is. And coherent sheaves on the functor can be described as coherent sheaves on the scheme, by descent theory.

7. Groupoid schemes

For group actions of an affine algebraic group G on a scheme X we will have the following informal dictionary

$$\begin{array}{ccc} X \text{ a principal } G-\text{bundle} & \Longleftrightarrow & [X/G] \text{ a scheme} \\ G \text{ acts freely on } X & \Longleftrightarrow & [X/G] \text{ an algebraic space} \\ \begin{array}{c} G \text{ acts on } X \text{ with fin. red. stabilizers} \\ (\text{e.g. if } G \text{ finite}) \end{array} & \Longleftrightarrow & [X/G] \text{ a Deligne}-\text{Mumford stack} \\ G \text{ acts on } X & \Longleftrightarrow & [X/G] \text{ an Artin stack} \\ G \text{ acts properly on } X & \Longleftrightarrow & [X/G] \text{ a separated Artin stack} \end{array}$$

A principal G-bundle structure is a locally trivial free group action with respect to a specified Grothendieck topology. It is a theorem of Grothendieck that if a *finite* group G acts freely on a scheme X , then the algebraic space $[X/G]$ is a scheme.

Definition 7.1. A *groupoid scheme* is a pair of schemes

$$R, U$$

and (surjective) morphisms

$$s, t : R \to U,$$

called respectively "source" and "target," with an identity

$$e : U \to R,$$

a multiplication

$$m : R \times_U R \to R,$$

and an inverse

$$i : R \to R$$

satisfying the following five properties:

1. Identity inverts both s and t:

$$\begin{array}{ccc} U & \xrightarrow{e} & R \\ & \searrow^{id_U} & \downarrow s \\ & & U \end{array} \quad \text{and} \quad \begin{array}{ccc} U & \xrightarrow{e} & R \\ & \searrow^{id_U} & \downarrow t \\ & & U \end{array} .$$

2. Multiplication is compatible with both s and t:

$$\begin{array}{ccc} R \times_U R & \xrightarrow{m} & R \\ \downarrow p_1 & & \downarrow s \\ R & \xrightarrow{s} & U \end{array} \quad \text{and} \quad \begin{array}{ccc} R \times_U R & \xrightarrow{m} & R \\ \downarrow p_2 & & \downarrow t \\ R & \xrightarrow{t} & U \end{array} .$$

3. Associativity: Commutative diagram

$$\begin{array}{ccc} R \times_U R \times_U R & \xrightarrow{id_R \times m} & R \times_U R \\ \downarrow m \times id_R & & \downarrow m \\ R \times_U R & \xrightarrow{m} & U \end{array} .$$

4. Unit condition: Commutative diagrams

$$\begin{array}{ccc} R & \xrightarrow{(e\circ s, id_R)} & R\times_U R \\ & \searrow^{id_R} & \downarrow m \\ & & R \end{array} \quad \textit{and} \quad \begin{array}{ccc} R & \xrightarrow{(e\circ t, id_R)} & R\times_U R \\ & \searrow^{id_R} & \downarrow m \\ & & R \end{array}.$$

5. Inverse:

$$\begin{aligned} i\circ i &= id_R \\ s\circ i &= t \\ t\circ i &= s \end{aligned}$$

and commutative diagrams

$$\begin{array}{ccc} R & \xrightarrow{(i,id_R)} & R\times_U R \\ \downarrow s & & \downarrow m \\ U & \xrightarrow{e} & R \end{array} \quad \textit{and} \quad \begin{array}{ccc} R & \xrightarrow{(i,id_R)} & R\times_U R \\ \downarrow t & & \downarrow m \\ U & \xrightarrow{e} & R \end{array}.$$

The groupoid scheme is etale, smooth, etc., if the seed morphisms s and t are etale, smooth, etc.

We should mention, in connection with the property 3 above, that there is a natural isomorphism

$$R\times_U R\times_R R\times_U R \cong (R\times_U R)\times_U R \cong R\times_U (R\times_U R)$$

coming from the fact that $R\times_U R\times_U R$ sits uniquely in the upper left and corner of the following fibered product diagram

$$\begin{array}{ccccc} & & R\times_U R & \xrightarrow{p_2} & R \\ & & \downarrow p_1 & & \downarrow s \\ R\times_U R & \xrightarrow{p_2} & R & \xrightarrow{t} & U \\ \downarrow p_1 & & \downarrow s & & \\ R & \xrightarrow{t} & U & & \end{array}.$$

Example 7.1. Our first example is a groupoid $\mathfrak{C}$, that is, a category whose only morphisms are isomorphisms:

$$\begin{aligned} Obj(\mathfrak{C}) &= U \\ Mor(\mathfrak{C}) &= R \end{aligned}$$

with

$$\begin{gathered} s: R\to U \\ (f: C_1\to C_2)\mapsto C_1 \end{gathered}$$

and

$$\begin{gathered} s: R\to U \\ (f: C_1\to C_2)\mapsto C_2 \end{gathered}$$

and

$$\begin{gathered} e: U\to R \\ C\mapsto id_C \end{gathered}$$

and

$$m(f\times_U g) = g\circ f$$

and

$$i(f) = f^{-1}.$$

Example 7.2. Our second example is an algebraic group action, that is, a scheme U with a left action

$$\sigma : G \times U \to U$$

by an algebraic group G. We put

$$R = G \times U,\ s = \sigma,\ t = p_2.$$

To define m first notice that the domain is naturally just

$$G \times G \times U$$

because the latter space fits uniquely into the upper left-hand corner of the fiber diagram

$$\begin{array}{ccc} G \times G \times U & \xrightarrow{p_{23}} & G \times U \\ \downarrow \sigma_{23} & & \downarrow \sigma \\ G \times U & \xrightarrow{p_2} & U \end{array}$$

where

$$\sigma_{23}(g, h, u) = (g, h \cdot u).$$

Then

$$(g, h, u) \leftrightarrow (g, h \cdot u) \times_U (h, u)$$

and

$$\begin{array}{c} m : G \times G \times U \to G \times U \\ (g, h, u) \mapsto (gh, u) \end{array}$$

and

$$i(g, u) = \left(g^{-1}, u\right).$$

Example 7.3. We begin with a morphism of schemes

$$f : T \to S$$

and let

$$\begin{array}{rcl} R & = & T' = T \times_S T \\ U & = & T \\ s & = & \pi_1 \\ t & = & \pi_2 \\ e & = & diag. : T \to T \times_S T. \end{array}$$

Again to define m, we notice via the cartesian diagram

$$\begin{array}{ccc} T \times_S T \times_S T & \xrightarrow{\pi_{23}} & T \times_S T \\ \downarrow \pi_{12} & & \downarrow \pi_1 \\ T \times_S T & \xrightarrow{\pi_2} & T \end{array}$$

we have

$$\begin{array}{rcl} T \times_S T \times_S T & \cong & (T \times_S T) \times_T (T \times_S T). \\ (t_1 \times t_2 \times t_3) & \leftrightarrow & (t_1 \times t_2) \times_T (t_2 \times t_3) \end{array}$$

So we can define

$$\begin{array}{rcl} m & = & \pi_{13} \\ p_1 & = & \pi_{12} \\ p_2 & = & \pi_{23}. \end{array}$$

Then, for example, 2 of Definition 7.1 becomes the cartesian diagrams

$$\begin{array}{ccc} T\times_S T\times_S T & \xrightarrow{\pi_{13}} & T\times_S T \\ \downarrow \pi_{12} & & \downarrow s \\ T\times_S T & \xrightarrow{s} & T \end{array} \quad and \quad \begin{array}{ccc} T\times_S T\times_S T & \xrightarrow{\pi_{23}} & T\times_S T \\ \downarrow \pi_{13} & & \downarrow t \\ T\times_S T & \xrightarrow{t} & T \end{array}.$$

And regarding 3 we have

$$\begin{array}{ccc} T\times_S T\times_S T\times_S T & \xrightarrow{\pi_{124}} & T\times_S T\times_S T \\ \downarrow \pi_{134} & & \downarrow \pi_{13} \\ T\times_S T\times_S T & \xrightarrow{\pi_{13}} & T\times_S T \end{array}$$

and for 4 we have

$$\begin{array}{ccc} T\times_S T & \xrightarrow{(e\circ s, id_R)} & T\times_S T\times_S T \\ & \searrow^{id_R} & \downarrow m \\ & & T\times_S T \end{array} \quad and \quad \begin{array}{ccc} T\times_S T & \xrightarrow{(e\circ t, id_R)} & T\times_S T\times_S T \\ & \searrow^{id_R} & \downarrow m \\ & & T\times_S T \end{array}.$$

Finally define

$$\begin{array}{c} i: T\times_S T \to T\times_S T. \\ (t_1,t_2)\mapsto(t_2,t_1) \end{array}$$

Example 7.4. This is the example that we will work with subsequently in these notes. Notice that the formalism of Example 7.3 works for any fibered category F with a functor

$$\underline{U}\to F$$

such that

$$\underline{U}\times_F\underline{U}$$

is representable, that is, is given by a functor

$$\underline{R}.$$

So we can form a category whose objects are

$$R\rightrightarrows U := (U,R,s,t,e,m,i)$$

and whose morphisms

$$(R'\rightrightarrows U')\to(R\rightrightarrows U)$$

are commutative diagrams

$$\begin{array}{ccc} R' & \xrightarrow{\Psi} & R \\ s'\downdownarrows t' & & s\downdownarrows t \\ U' & \xrightarrow{\psi} & U \end{array}$$

and

$$\begin{array}{ccc} R' & \xrightarrow{\Psi} & R \\ \uparrow e' & & \uparrow e \\ U' & \xrightarrow{\psi} & U \end{array}$$

and

$$\begin{array}{ccc} R' \times_{U'} R' & \xrightarrow{\Psi} & R \times_U R \\ \downarrow m' & & \downarrow m \\ R' & \xrightarrow{\Psi} & R \end{array}$$

and

$$\begin{array}{ccc} R' & \xrightarrow{\Psi} & R \\ \downarrow i' & & \downarrow i \\ R' & \xrightarrow{\Psi} & R \end{array}$$

8. Basic stack construction

8.1. Category built on a target groupoid scheme.

Proposition 8.1. *Every Deligne-Mumford stack comes from an etale groupoid scheme. In fact we have a functor*

$$(R \rightrightarrows U) \to [R \rightrightarrows U]$$

from the category of groupoid schemes to the category of stacks.

We will describe the process of associating a DM-stack to a groupoid scheme in several steps. Fix a groupoid scheme

$$R \rightrightarrows U.$$

For each morphism of schemes

$$T \to S$$

define a category $F_{T\to S}$ by

$$Obj\left(F_{T\to S}\right) = Mor\left((T_{\times S}T) \rightrightarrows (R \rightrightarrows U)\right),$$

that is, commutative diagrams

$$\begin{array}{ccc} T_{\times S}T & \xrightarrow{\Psi} & R \\ \pi_1 \downdownarrows \pi_2 & & s \downdownarrows t \\ T & \xrightarrow{\psi} & U \end{array},$$

and

$$Mor\left(F_{T\to S}\right) = \left\{ \begin{array}{ccc} T_{\times S}T & \xrightarrow{\Psi_1} & R \\ \pi_1 \downdownarrows \pi_2 & & s \downdownarrows t \\ T & \xrightarrow{\psi_1} & U \end{array} \Longrightarrow \begin{array}{ccc} T_{\times S}T & \xrightarrow{\Psi_2} & R \\ \pi_1 \downdownarrows \pi_2 & & s \downdownarrows t \\ T & \xrightarrow{\psi_2} & U \end{array} \right\}$$

given by a map

$$\begin{array}{ccc} & & \begin{array}{ccc} T_{\times S}T & \xrightarrow{\Psi_1} & R \\ \pi_1 \downdownarrows \pi_2 & & s \downdownarrows t \\ T & \xrightarrow{\psi_1} & U \end{array} \\ & \swarrow \alpha & \\ \begin{array}{ccc} T_{\times S}T & \xrightarrow{\Psi_2} & R \\ \pi_1 \downdownarrows \pi_2 & & s \downdownarrows t \\ T & \xrightarrow{\psi_2} & U \end{array} & & \end{array} \tag{27}$$

for which

$$\begin{aligned} s \circ \alpha &= \psi_1 \\ t \circ \alpha &= \psi_2 \end{aligned}$$

and

$$m \circ (\alpha \circ \pi_1, \Psi_2) = m \circ (\Psi_1, \alpha \circ \pi_2)$$

We compose morphisms as follows. Given

$$\alpha, \beta : T \to R$$

such that

$$t \circ \alpha = s \circ \beta,$$

define

$$\beta \circ \alpha = m(\alpha, \beta).$$

We then check that the composition

$$(\psi_1, \Psi_1) \xrightarrow{\alpha} (\psi_2, \Psi_2) \xrightarrow{\beta} (\psi_3, \Psi_3)$$

given by

$$m(\alpha, \beta)$$

is a morphism. First of all

$$\begin{aligned} s \circ m(\alpha, \beta) &= s \circ \alpha = \psi_1 \\ t \circ m(\alpha, \beta) &= t \circ \beta = \psi_2 \end{aligned}$$

and then

$$\begin{aligned} m(m(\alpha, \beta) \circ \pi_1, \Psi_3) &= m(\alpha \circ \pi_1, m(\beta \circ \pi_1, \Psi_3)) \\ &= m(\alpha \circ \pi_1, m(\Psi_2, \beta \circ \pi_2)) \\ &= m(m(\alpha \circ \pi_1, \Psi_2), \beta \circ \pi_2) \\ &= m(m(\Psi_1, \alpha \circ \pi_2), \beta \circ \pi_2) \\ &= m(\Psi_1, m(\alpha, \beta) \circ \pi_2). \end{aligned}$$

Proposition 8.2. *$F_{T \to S}$ is a groupoid, that is, a category all of whose morphisms are isomorphisms.*

Proof. First of all, there is an identity morphism

$$\begin{array}{ccccccc} T{\times_S}T & \xrightarrow{\Psi} & R & & T{\times_S}T & \xrightarrow{\Psi} & R \\ \pi_1 \downdownarrows \pi_2 & & s \downdownarrows t & \Longrightarrow & \pi_1 \downdownarrows \pi_2 & & s \downdownarrows t \\ T & \xrightarrow{\psi} & U & & T & \xrightarrow{\psi} & U \end{array}$$

given by

$$\alpha = e \circ \psi$$

since

$$s \circ e \circ \psi = t \circ e \circ \psi = \psi$$

and

$$m(e \circ \psi, \Psi) = \Psi = m(\Psi, e \circ \psi).$$

So we need to find an inverse for

$$\begin{array}{ccc} T_{\times S}T & \stackrel{\Psi_1}{\longrightarrow} & R \\ \pi_1 \downdownarrows \pi_2 & & s \downdownarrows t \\ T & \stackrel{\psi_1}{\longrightarrow} & U \end{array} \stackrel{\alpha}{\Longrightarrow} \begin{array}{ccc} T_{\times S}T & \stackrel{\Psi_2}{\longrightarrow} & R \\ \pi_1 \downdownarrows \pi_2 & & s \downdownarrows t \\ T & \stackrel{\psi_2}{\longrightarrow} & U \end{array}.$$

To do this, we define

$$j : R \times_U R \stackrel{(i \circ p_2, i \circ p_1)}{\longrightarrow} R \times_U R$$

and note that

$$m\left(m\left(a,b\right),\left(m\circ j\right)\left(a,b\right)\right)=m\left(m\left(a,b\right),m\left(i\left(b\right),i\left(a\right)\right)\right)=e\circ s=e\circ t$$

and so, by the associativity property 3 of groupoid schemes,

$$\begin{array}{rcl} m\left(\left(\left(i\circ m\right)\left(a,b\right),m\left(a,b\right)\right),\left(m\circ j\right)\left(a,b\right)\right) & = & m\left(\left(i\circ m\right)\left(a,b\right),\left(m,m\circ j\right)\left(a,b\right)\right) \\ \left(m\circ j\right)\left(a,b\right) & = & \left(i\circ m\right)\left(a,b\right). \end{array}$$

If we put

$$\beta = i \circ \alpha,$$

then

$$\begin{array}{rcl} s\circ i\circ\alpha & = & t\circ\alpha=\psi_2 \\ t\circ i\circ\alpha & = & s\circ\alpha=\psi_1 \end{array}$$

and

$$\begin{array}{rcl} m\circ\left(i\circ\alpha\circ\pi_1,\Psi_1\right) & = & i\circ m\circ j\left(i\circ\alpha\circ\pi_1,\Psi_1\right) \\ & = & i\circ m\circ\left(i\circ\Psi_1,\alpha\circ\pi_1\right) \\ & = & i\circ m\circ\left(\Psi_1\circ i,\alpha\circ\pi_2\circ i\right) \\ & = & i\circ m\circ\left(\Psi_1,\alpha\circ\pi_2\right)\circ i \\ & = & i\circ m\circ\left(\alpha\circ\pi_1,\Psi_2\right)\circ i \\ & = & i\circ m\circ\left(\alpha\circ\pi_2,i\circ\Psi_2\right) \\ & = & i\circ m\circ\left(\Psi_2,\alpha\circ\pi_2\right). \end{array}$$

So

$$\beta = i \circ \alpha$$

is the inverse morphism. □

8.2. **Pullback and category** $F = [R \rightrightarrows U]$**.** Given a commutative diagram of morphisms of schemes

$$\begin{array}{ccc} T' & \to & S' \\ \downarrow & & \downarrow \\ T & \to & S \end{array}$$

there is an obvious pullback functor

$$F_{T\to S} \to F_{T'\to S'}.$$

Proposition 8.3. *Given a groupoid scheme*

$$R \rightrightarrows U$$

there exists a fibered category

$$F = [R \rightrightarrows U]$$

with

$$F_S = \lim_{\leftarrow} F_{T \to S}$$

where the limit is taken over all etale covers

$$T \to S,$$

and, for

$$S_1 \to S_2$$

we have

$$Mor\left(\lim_{\leftarrow} F_{T_1 \to S_1}, \lim_{\leftarrow} F_{T_2 \to S_2}\right)$$

induced by maps

$$\alpha : T_1 \times_{S_2} T_2 \to R$$

defining morphisms from the pullback of

$$\begin{array}{ccc} T_1 \times_{S_1} T_1 & \xrightarrow{\Psi_1} & R \\ \pi_{1,1} \downdownarrows \pi_{1,2} & & s \downdownarrows t \\ T_1 & \xrightarrow{\psi_1} & U \end{array}$$

by

$$\begin{array}{ccc} T_1 \times_{S_1} T_2 & \xrightarrow{\pi_1} & T_1 \\ \downarrow & & \downarrow \\ S_1 \times_{S_2} S_2 & \xrightarrow{\pi_1} & S_1 \end{array}$$

to the pullback of

$$\begin{array}{ccc} T_2 \times_{S_2} T_2 & \xrightarrow{\Psi_2} & R \\ \pi_{2,1} \downdownarrows \pi_{2,2} & & s \downdownarrows t \\ T_2 & \xrightarrow{\psi_2} & U \end{array}$$

by

$$\begin{array}{ccc} T_1 \times_{S_1} T_2 & \xrightarrow{\pi_2} & T_2 \\ \downarrow & & \downarrow \\ S_1 \times_{S_2} S_2 & \xrightarrow{\pi_2} & S_2 \end{array}.$$

Example 8.1. Let

$$g : U \to X$$

be a covering. Then given a covering

$$f : T' \to T$$

and an object

$$\begin{array}{ccc} T' \times_T T' & \xrightarrow{\Psi} & U \times_X U \\ \pi_1 \downdownarrows \pi_2 & & s \downdownarrows t \\ T' & \xrightarrow{\psi} & U \\ \downarrow f & & \downarrow g \\ T & \dashrightarrow & X \end{array} .$$

Since

$$g \circ \psi \circ \pi_1 = g \circ \psi \circ \pi_2$$

we have

$$h : T \to X$$

induced by descent as in the proof of Proposition 4.2. (Notice that we do not need that g be a cover for this.) Now the only morphism

$$\begin{array}{ccc} T' \times_T T' & \xrightarrow{\Psi_1} & U \times_X U \\ \pi_1 \downdownarrows \pi_2 & & s \downdownarrows t \\ T' & \xrightarrow{\psi_1} & U \\ \downarrow f & & \downarrow g \\ T & \dashrightarrow & X \end{array} \overset{\alpha}{\Rightarrow} \begin{array}{ccc} T' \times_T T' & \xrightarrow{\Psi_2} & U \times_X U \\ \pi_1 \downdownarrows \pi_2 & & s \downdownarrows t \\ T' & \xrightarrow{\psi_2} & U \\ \downarrow f & & \downarrow g \\ T & \dashrightarrow & X \end{array}$$

which works is over a given diagram

$$\begin{array}{ccccc} T & & & & T \\ & {\scriptstyle h_1} \searrow & & \swarrow {\scriptstyle h_2} & \\ & & X & & \end{array}$$

is

$$\alpha = (\psi_1, \psi_2)$$

because of the conditions

$$\begin{aligned} s \circ \alpha &= \psi_1 \\ t \circ \alpha &= \psi_2 . \end{aligned}$$

Then

$$[U \times_X U \rightrightarrows U] = \underline{X}.$$

Referring to Examples 1.8 and 7.2 we also have:

Proposition 8.4. *If G is an affine algebraic group and U has a group action σ, then*

$$[G \times U \rightrightarrows U] = [U/G] .$$

Definition 8.1. If a stack

$$[U/G]$$

is such that

$$G \times U \xrightarrow{(s,t)} U \times U \tag{28}$$

is a closed immersion, we call the stack an *algebraic space*. Notice that in this case the action must be free in order that (28) be injective.

It will turn out that, if G acts on U with finite stabilizers, then

$$[U/G]$$

is (isomorphic to) a Deligne-Mumford stack.

8.3. **Dictionary.** For stacks induced from groupoid schemes we will have the following informal dictionary:

$$\begin{array}{ccc} \textit{groupoid scheme, } s,t \textit{ flat} & \Longrightarrow & \textit{stack} \\ s,t \textit{ smooth, } R \stackrel{(s,t)}{\longrightarrow} U\times U \textit{ immersion} & \Longleftrightarrow & [R\rightrightarrows U] \textit{ a locally separated algebraic space} \\ s,t \textit{ etale} & \Longrightarrow & [R\rightrightarrows U] \textit{ a Deligne}-\textit{Mumford stack} \\ s,t \textit{ smooth} & \Longleftrightarrow & [R\rightrightarrows U] \textit{ an Artin stack} \end{array}.$$

Again it is a theorem of Grothendieck that, it s and t are finite and etale, and

$$R \stackrel{(s,t)}{\longrightarrow} U \times U$$

is an immersion, the Deligne-Mumford stack $[R \rightrightarrows U]$ is actually a scheme.

9. $M_{1,1}$ as a Deligne-Mumford stack

We next give *the* basic example of a stack in the etale topology. This is the moduli stack of elliptic curves

$$M_{1,1}$$

that is, genus one curves with a marked point.

$$\begin{array}{rcl} Obj\,(M_{1,1}) & = & \left\{ \begin{array}{c} E_S \\ \pi \downarrow\uparrow e_S \\ S \end{array} : E_S \textit{ smooth genus one over } S \in Obj\,(\mathfrak{S}) \right\} \\ Mor\,(M_{1,1}) & = & \left\{ \begin{array}{ccc} E_S & \to & E_{S'} \\ \pi \downarrow\uparrow e_S & & \pi' \downarrow\uparrow e_{S'} \\ S & \to & S' \end{array} \textit{ doubly commutative} \right\}. \end{array}$$

$M_{1,1}$ is a category fibered in groupoids. In particular, over

$$(M_{1,1})_{point}$$

is the category whose objects are elliptic curves and whose morphisms are isomorphisms of those curves (preserving the distinguished zero, of course). Thus

$$Mor\,(E_{point}, E_{point})$$

is in general $\mathbb{Z}/2\mathbb{Z}$ but can be larger if E_{point} has non-trivial automorphisms.

We realize this stack as

$$[R \rightrightarrows U]$$

as above where $R \rightrightarrows U$ is a groupoid scheme. To do this, let

$$U = \mathbb{A}^1 - \{0,1\}$$

and let

$$E_U \subset \mathbb{P}^2 \times U$$

be given by

$$E_U = \left\{(x,y,z,\lambda) : y^2 z = x\,(x-z)\,(x-\lambda z)\right\} \tag{29}$$

with distinguished point

$$(0,1,0).$$

Next, letting

$$\Sigma = e_S(S)$$

define the rank-2 vector bundle

$$\pi_* \mathcal{O}_E(2\Sigma).$$

Then

$$f : E \to P := \mathbb{P}\left((\pi_* \mathcal{O}_E(2\Sigma))^\vee\right)$$

is finite and flat (by Grothendieck's criterion for flatness). Then the line-bundle section

$$df \in Hom\left(T_{E/S}, f^* T_{P/S}\right)$$

has four simple zeros giving an etale divisor Δ of degree 4 over S. Clearly

$$\Delta = \Sigma \amalg \Delta'$$

where Δ' is etale of degree 3. Define a stack $\tilde{M}_{1,1}$ by

$$Obj\left(\tilde{M}_{1,1}\right) = \{(E/S \in M_{1,1}, \phi)\}$$

where ϕ is an isomorphism between Δ' and an ordered set

$$\Sigma_1 \amalg \Sigma_2 \amalg \Sigma_3$$

of three disjoint copies of Σ. We have sections

$$e_{S,i} : S \to \Sigma_i.$$

Proposition 9.1. *There is a functor*

$$\tilde{M}_{1,1} \to M_{1,1}$$

which is etale of order 6*.*

Proof. For

$$e_{S,1} : \underline{S} \to M_{1,1}$$

we find the fibered product

$$\tilde{M}_{1,1} \times_{M_{1,1}} \underline{S}$$

as the stack obtained by removing the (generalized) diagonal from

$$\Delta' \times_S \Delta' \times_S \Delta'.$$

This stack is etale of order 6 over $\underline{S}$. □

Proposition 9.2. *The stack whose objects are families*

$$P \to S$$

of smooth rational curves together with an ordered set of four disjoint sections

$$\Sigma, \Sigma_1, \Sigma_2, \Sigma_3 \subset P$$

is isomorphic to

$$\underline{U} = \underline{\mathbb{P}^1 - \{0,1,\infty\}}.$$

Proof. Isomorphisms of $\mathbb{P}^1$ are uniquely determined by 3 points. □

Thus via

$$f : E \to P := \mathbb{P}\left(\left(\pi_* \mathcal{O}_E(2\Sigma)\right)^\vee\right)$$

we obtain a morphism

$$\sigma : \tilde{M}_{1,1} \to \underline{U}.$$

On the other hand, (29) gives a morphism

$$\tau : \underline{U} \to \tilde{M}_{1,1}.$$

This last is etale of degree 2 as can be seen by defining

$$\begin{array}{ccc} \underline{S} & \to & \tilde{M}_{1,1} \\ S & \to & \Sigma \times_S \Sigma_1 \times_S \Sigma_2 \times_S \Sigma_3 \end{array}$$

and obtaining that

$$\underline{U} \times_{\tilde{M}_{1,1}} \underline{S} \to \underline{S}$$

is etale of degree 2 corresponding to the choice of

$$\sqrt{x(x-1)(x-\lambda)}.$$

Thus the obvious functor

$$\underline{U} \to M_{1,1}$$

given by

$$\left(S \xrightarrow{f} U\right) \to \left(f^* E_U\right)/S$$

factors as

$$\underline{U} \to \tilde{M}_{1,1} \to M_{1,1}$$

where the first morphism is etale of degree 2 and the second is etale of degree 6. Recalling from Example 1.2 that, for the group scheme $G = \mu_2 = \{\pm 1\}$ we have the stack

$$BG \to \mathfrak{S}$$

where

$$F \to S$$

is a principle μ-bundle, that is, we have an action

$$\begin{array}{ccccc} F \times \mu_2 & & \longrightarrow & & F \\ & \searrow & & \swarrow & \\ & & S & & \end{array}.$$

We have

$$\begin{array}{rcl} Obj\left(B\mu_2\right) & = & \{F_S\} \\ Mor\left(B\mu_2\right) & = & Hom_{B\mu}\left(F_S, F'_{S'}\right) \\ & = & \{F_S \longleftrightarrow S \times_{S'} F'_{S'}\}. \end{array}$$

Then the "section" $\sigma : \tilde{M}_{1,1} \to \underline{U}$ defined above allows the identification

$$\underline{U} \times_\tau \underline{U} = B\mu_2$$

and so the isomorphism

$$\tilde{M}_{1,1} = \underline{U} \times B\mu_2.$$

More concretely, given

$$\begin{array}{ccc} \Delta & \subset & S \times \mathbb{P}^1 \\ \updownarrow & & \updownarrow \\ S \times \{0, 1, \infty, \lambda\} & & S \times \mathbb{P}^1 \end{array}$$

the branched double covers given by the inverse image of the tautological section under a squaring map

$$L \to L^2 = \mathcal{O}_{S \times \mathbb{P}^1}(\Delta) \tag{30}$$

comprise the objects of $\tilde{M}_{1,1}$ and are classified by the non-trivial objects of $B\mu_2$. Notice that the structure sheaf of the branched double cover associated to a choice of L is given by

$$\mathcal{O}_{S \times \mathbb{P}^1} \oplus L^{-1}$$

with multiplication determined by the choice of preimage in (30):

$$(f, \alpha) \cdot (f', \alpha') = (ff' + \alpha\alpha',\ f\alpha' + f'\alpha)$$

where the section $\alpha\alpha'$ of L^{-2} is identified with its image under the inclusion

$$L^{-2} = \mathcal{O}_{S \times \mathbb{P}^1}(-\Delta) \subseteq \mathcal{O}_{S \times \mathbb{P}^1}.$$

This leads us to the computation of

$$R = U \times_{M_{1,1}} U.$$

To compute R, we must understand all cartesian diagrams

$$\begin{array}{ccc} & & E_U \\ & & \downarrow \\ E_U & \to & E_V \end{array} \Rightarrow \begin{array}{ccc} & & U \\ & & \downarrow^f \\ U & \xrightarrow{f'} & V \end{array}.$$

Such a diagram implies

$$E_{\lambda'} = E_{f'(\lambda')} = E_{f(\lambda)} = E_{\lambda}.$$

Also deformations of families of isomorphisms of elliptic curves are unobstructed, so that, given

$$(\lambda_0, \lambda_0') \in U \times U$$

and

$$\alpha_0 : E_{\lambda_0'} \cong E_{\lambda_0} \tag{31}$$

there is a unique, smooth germ of a curve $D \subseteq U \times U$ passing through (λ_0, λ_0') such that the isomorphism (31) extends uniquely to an isomorphism

$$E_{\lambda'} \cong E_{\lambda}$$

for all $(\lambda, \lambda') \in D$ and any scheme in $U \times U$ over which (31) extends is locally contained in D. Using this we check:

Lemma 9.3. *Let*

$$(\lambda, \lambda') \in U \times_{M_{1,1}} U \to U \times U.$$

Any isomorphism

$$\begin{array}{rcl} \alpha & : & E_\lambda \cong E_{\lambda'} \\ (x, y) & \mapsto & (x', y') \end{array}$$

has the form

$$\begin{array}{rcl} x' & = & u^2 x + r \\ y' & = & u^3 y \end{array}$$

for some $u \neq 0$ and some r.

Therefore we have six cases:

λ'	$a = u^2$	$d = u^3$	r
λ	1	± 1	0
$\frac{1}{\lambda}$	$\frac{1}{\lambda}$	$\frac{\pm 1}{\sqrt{\lambda^3}}$	0
$1 - \frac{1}{\lambda}$	$-\frac{1}{\lambda}$	$\frac{\pm i}{\sqrt{\lambda^3}}$	1
$1 - \lambda$	-1	$\pm i$	1
$\frac{\lambda}{\lambda - 1}$	$\frac{1}{1-\lambda}$	$\frac{\pm 1}{\sqrt{(1-\lambda)^3}}$	λ'
$\frac{1}{1-\lambda}$	$\frac{-1}{1-\lambda}$	$\frac{\pm i}{\sqrt{(1-\lambda)^3}}$	λ'

Proof. Put

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = A(\lambda, \lambda') \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Since

$$(0, 1, 0)$$

and

$$z = 0$$

are preserved, we can assume

$$A = \begin{pmatrix} a & 0 & r \\ c & d & e \\ 0 & 0 & 1 \end{pmatrix}$$

Thus

$$\begin{array}{rcl} x' & = & ax + r \\ y' & = & cx + dy + e. \end{array}$$

Since points of order two must correspond under the isomorphism,

$$\begin{array}{rcl} c \cdot 1 + e & = & 0 \\ c \cdot \lambda + e & = & 0 \end{array}$$

so in fact

$$\begin{array}{rcl} x' & = & ax + r \\ y' & = & d \cdot y. \end{array}$$

But

$$\begin{aligned}(d\cdot y)^2 &= (ax+r)(ax+r-1)(ax+r-\lambda')\\ &= d^2\cdot x(x-1)(x-\lambda)\end{aligned}$$

so that

$$\begin{aligned}a^3 &= d^2\\ a^2(3r-1-\lambda') &= d^2(-1-\lambda)\\ a(r(2r-1-\lambda')+(r-1)(r-\lambda')) &= d^2\lambda\\ r(r-1)(r-\lambda') &= 0.\end{aligned}$$

Now we can put $u=\frac{d}{a}$ is our scheme lies in the smooth scheme D defined above so we have

(32)
$$\begin{aligned}r &= 0,1,\lambda'\\ (3r-1-\lambda') &= u^2(-1-\lambda)\\ r(2r-1-\lambda')+(r-1)(r-\lambda') &= au^2\lambda.\end{aligned}$$

Case: $r=0$: Either

$$\begin{aligned}a\lambda &= \lambda',\ a=1\\ \lambda' &= \lambda\end{aligned}$$

or

$$\begin{aligned}a\lambda &= 1,\ a=\lambda'\\ \lambda' &= \frac{1}{\lambda}\end{aligned}$$

so that (32) becomes

(33)
$$\begin{aligned}\lambda' &= \lambda\\ u^2 &= 1\end{aligned}$$

or

(34)
$$\begin{aligned}\lambda' &= \frac{1}{\lambda}\\ u^2 &= \frac{1}{\lambda}.\end{aligned}$$

Case $r=1$: Either

$$\begin{aligned}a\cdot 1+1 &= \lambda',\ a\cdot\lambda+1=0\\ \lambda' &= 1-\frac{1}{\lambda}\end{aligned}$$

or

$$\begin{aligned}a\cdot 1+1 &= 0,\ a\cdot\lambda+1=\lambda'\\ \lambda' &= 1-\lambda\end{aligned}$$

so that (32) becomes

(35)
$$\lambda' = 1-\frac{1}{\lambda}$$

(36)
$$u^2 = -\frac{1}{\lambda}$$

or

(37)
$$\begin{aligned} \lambda' &= 1-\lambda \\ u^2 &= -1. \end{aligned}$$

Case $r=\lambda'$: Either

$$\begin{aligned} a\cdot 1+\lambda' &= 1,\ a\cdot\lambda+\lambda'=0 \\ \lambda' &= \frac{\lambda}{\lambda-1} \end{aligned}$$

or

$$\begin{aligned} a\cdot 1+\lambda' &= 0,\ a\lambda+\lambda'=1 \\ \lambda' &= \frac{1}{1-\lambda}. \end{aligned}$$

so that (32) becomes either

(38)
$$\begin{aligned} \lambda' &= \frac{\lambda}{\lambda-1} \\ u^2 &= \frac{1}{1-\lambda} \end{aligned}$$

or

(39)
$$\begin{aligned} \lambda' &= \frac{1}{1-\lambda} \\ u^2 &= \frac{-1}{1-\lambda} \end{aligned}$$

□

Lemma 9.3 allows us to list the components of R as follows:

(40)

λ'	x'	y'	*comp R*
λ	x	$\pm y$	$Spec\left(\frac{k[\lambda,u]}{u^2-1}\right)$
$\frac{1}{\lambda}$	$\frac{1}{\lambda}x$	$\frac{\pm 1}{\sqrt{\lambda^3}}y$	$Spec\left(\frac{k[\lambda,u]}{\lambda u^2-1}\right)$
$1-\frac{1}{\lambda}$	$-\frac{1}{\lambda}x+1$	$\frac{\pm i}{\sqrt{\lambda^3}}y$	$Spec\left(\frac{k[\lambda,u]}{\lambda u^2+1}\right)$
$1-\lambda$	$-x+1$	$\pm iy$	$Spec\left(\frac{k[\lambda,u]}{u^2+1}\right)$
$\frac{\lambda}{\lambda-1}$	$\frac{1}{1-\lambda}x+\frac{\lambda}{\lambda-1}$	$\frac{\pm 1}{\sqrt{(1-\lambda)^3}}y$	$Spec\left(\frac{k[\lambda,u]}{u^2-u^2\lambda-1}\right)$
$\frac{1}{1-\lambda}$	$\frac{-1}{1-\lambda}x+\frac{1}{1-\lambda}$	$\frac{\pm i}{\sqrt{(1-\lambda)^3}}y$	$Spec\left(\frac{k[\lambda,u]}{u^2-u^2\lambda+1}\right)$

so that four of the components

$$R_{\lambda,1}, R_{\lambda,-1}, R_{1-\lambda,i}, R_{1-\lambda,-i}$$

map isomorphically to U via λ and λ'. But for the other four u is a uniformizing parameter and we have

$$\begin{aligned}
R_{\frac{1}{\lambda}} &\to U : u \mapsto \lambda = \frac{1}{u^2},\ \lambda' = u^2 \\
R_{1-\frac{1}{\lambda}} &\to U : u \mapsto \lambda = \frac{-1}{u^2},\ \lambda' = 1 + u^2 \\
R_{\frac{\lambda}{\lambda-1}} &\to U : u \mapsto \lambda = 1 - \frac{1}{u^2}, 1 - u^2 \\
R_{\frac{1}{1-\lambda}} &\to U : u \mapsto \lambda = 1 + \frac{1}{u^2}, \lambda' = -u^2.
\end{aligned}$$

Lemma 9.4.

$$U \times_{M_{1,1}} U \rightrightarrows U$$

is a groupoid scheme with

$$\begin{aligned}
s &= \lambda \\
t &= \lambda'.
\end{aligned}$$

Proof. First

$$e : U \to R$$

is just the identity map to the component

$$\lambda = \lambda',\ u = 1.$$

Also

$$m : R \times_U R \to R$$

is just the map induced by the composition of isomorphisms

$$(x, y) \mapsto (x', y') \mapsto (x'', y'')\,.$$

Similarly the inverse

$$i : R \to R$$

just sends

$$(x, y) \mapsto (x', y')$$

to

$$(x', y') \mapsto (x, y)\,.$$

For example

$$\begin{aligned}
x' &= \frac{1}{1-\lambda}x + \frac{\lambda}{\lambda - 1} \\
y' &= \frac{1}{\sqrt{(1-\lambda)^3}}y
\end{aligned}$$

composes with

$$\begin{aligned}
x'' &= \frac{-1}{1-\lambda'}x' + \frac{1}{1-\lambda'} \\
y'' &= \frac{i}{\sqrt{(1-\lambda')^3}}y'
\end{aligned}$$

to give

$$\begin{aligned} x'' &= \frac{-1}{1-\frac{\lambda}{\lambda-1}}\left(\frac{1}{1-\lambda}x+\frac{\lambda}{\lambda-1}\right)+\frac{1}{1-\frac{\lambda}{\lambda-1}} \\ &= -x+1 \\ y'' &= \frac{i}{\sqrt{\left(1-\frac{\lambda}{\lambda-1}\right)^3}}y' = \frac{i}{\sqrt{\left(1-\frac{\lambda}{\lambda-1}\right)^3}}\frac{1}{\sqrt{(1-\lambda)^3}}y \\ &= iy. \end{aligned}$$

Also

$$\begin{aligned} x' &= -x+1 \\ y' &= iy \end{aligned}$$

has inverse

$$\begin{aligned} x' &= -x+1 \\ y' &= -iy \end{aligned}$$

which allows us to compute the inverse of the transformation

$$\begin{aligned} x' &= \frac{1}{1-\lambda}x+\frac{\lambda}{\lambda-1} \\ y' &= \frac{1}{\sqrt{(1-\lambda)^3}}y \end{aligned}$$

above. □

Notice that the stack

$$M_{1,1} \neq [U/G]$$

for any group G acting on a scheme U. If it were, we would infer that

$$R = G \times U.$$

However we do have a closely related stack

$$M_{1,1}[2]$$

of level-two structures on elliptic curves. Here morphisms must preserve distinguished sections

$$e, h, h' : S \to E_S$$

where h and h' go to distinct half-periods. Thus

$$R[2]$$

has only two components

$$R[2]_{\lambda,1}, R[2]_{\lambda,-1}$$

with each map to U given by the identity. There is a natural morphism of stacks

$$M_{1,1}[2] \to M_{1,1}.$$

On the other hand, for

$$\frac{M_{1,1}}{\pm 1}$$

obtained by equating two morphisms which differ by

$$(x, y) \longmapsto (x, -y)$$

we find that R now has six components, each isomorphic to U and

$$\frac{M_{1,1}}{\pm 1} = [U/S_3]$$

where

$$S_3 = \left\{ id, \frac{1}{id}, 1 - id, \frac{1}{1 - id}, \frac{id}{id - 1}, \frac{id - 1}{id} \right\}. \tag{41}$$

The map

$$U \to U/S_3$$

is given by

$$\lambda \mapsto -64 \frac{\left(\lambda^2 - \lambda + 1\right)^3}{\left(\lambda \left(\lambda - 1\right)\right)^2}$$

since this mapping is $6 - 1$ and invariant under the substitutions

$$\begin{aligned} \lambda &\mapsto \frac{1}{\lambda} \\ \lambda &\mapsto 1 - \lambda. \end{aligned}$$

Notice that S_3 acts freely except when

$$\begin{aligned} \lambda &= -1, 2, \frac{1}{2} \\ \lambda &= -\omega \end{aligned}$$

where

$$\omega = -\frac{1}{2} \pm \frac{\sqrt{-3}}{2}.$$

The isotropy groups for the action of S_3 are respectively

$$\begin{aligned} I_{-1} &= \mathbb{Z}/2\mathbb{Z} \\ I_{-\omega} &= \mathbb{Z}/3\mathbb{Z}. \end{aligned}$$

We will use all this to show that

$$Pic \frac{M_{1,1}}{\pm 1} = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}. \tag{42}$$

A line bundle on the stack

$$\frac{M_{1,1}}{\pm 1} = [U/S_3]$$

is just an S_3-invariant line bundle L_{-1} on U and L descends to a line bundle on

$$\mathbb{A}^1 = \frac{U}{S_3}$$

if and only if I_{-1} acts trivially on the fiber L_{-1} and $I_{-\omega}$ acts trivially on the fiber $L_{-\omega}$. Since all line bundles on $\mathbb{A}^1$ are trivial, the morphism

$$\begin{aligned} Pic \frac{M_{1,1}}{\pm 1} &\to \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \\ L &\mapsto \left(I_{-1} \to Aut\left(L_{-1}\right),\ I_{-\omega} \to Aut\left(L_{-\omega}\right)\right) \end{aligned} \tag{43}$$

is injective. To see that it is surjective, consider, for the Weierstrass family

$$\pi : E \to U$$

the bundle

$$\Lambda = \pi_* \omega_{E/U}.$$

Since

$$E_{-1} = \{y^2 = x^3 - x\}$$

the generator τ of I_{-1} is given by

$$\tau(x, y) = (-x, iy)$$

and so

$$\tau^* \left(\frac{dx}{y}\right) = i\frac{dx}{y}.$$

Similarly since

$$E_{-\omega} = \{y^2 = x^3 - 1\}$$

he generator σ of $I_{-\omega}$ is given by

$$\sigma(x, y) = (\omega x, -y)$$

and so

$$\tau^* \left(\frac{dx}{y}\right) = -\omega\frac{dx}{y}.$$

Thus

$$L = \Lambda^2$$

gives a well-defined S_3-invariant bundle on U which maps to the generator of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ under (43).

Said another way, we have $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}$ ways of assigning never-zero functions

$$f_\lambda, f_{1/\lambda}, f_{1-\lambda}, f_{1/1-\lambda}, f_{\lambda/\lambda-1}, f_{1-1/\lambda}$$

on U such that

$$(f_{k'} \circ k) \cdot f_k = f_{k' \circ k}. \tag{44}$$

Basically we get these by letting

$$f_\lambda$$

be any of the six functions in (41) and noticing that the rest of the assignments are determined by the cochain condition (44) .

Lemma 9.5.

$$PicM_{1,1} = \mathbb{Z}/12\mathbb{Z}.$$

Proof. We begin with the line bundles Λ_1 and Λ_2 induced by the two pull-backs of Λ to

$$R = U \times_{M_{1,1}} U \rightrightarrows U.$$

We have and isomorphism

$$\varphi : \Lambda_1 \to \Lambda_2$$

induced by (40) which satisfies the cocycle condition. However

$$\begin{array}{rcl} U & \to & \mathbb{A}^1 \\ \lambda & \mapsto & -64\dfrac{\left(\lambda^2-\lambda+1\right)^3}{\left(\lambda\left(\lambda-1\right)\right)^2} \end{array}$$

is not etale, so we can't argue that by etale descent there is an induced line bundle on $\mathbb{A}^1$.

Let

$$\tilde{I}_{-1} \subseteq Aut\left(L_{-1}\right)$$

be the group of automorphisms

$$\left(\frac{dx}{y}\right) \mapsto a\left(\frac{dx}{y}\right)$$

of

$$H^0\left(\omega_{E_{-1}}\right)$$

occurring on (40). that is, automorphisms of $H^0\left(\omega_{E_{-1}}\right)$ induced by isomorphisms

$$\begin{array}{ccc} E & \xrightarrow{\tilde{\alpha}} & E \\ \downarrow & & \downarrow \\ U & \xrightarrow{\alpha} & U \end{array}$$

such that

$$\alpha\left(-1\right) = -1.$$

From (40) we have

$$\tilde{I}_{-1} = \mathbb{Z}/4\mathbb{Z}.$$

Similarly

$$\tilde{I}_{-\omega} = \mathbb{Z}/6\mathbb{Z}.$$

We then consider the morphism

$$\begin{array}{rcl} PicM_{1,1} & \to & \mathbb{Z}/4\mathbb{Z}\times_{\mathbb{Z}/2\mathbb{Z}}\mathbb{Z}/6\mathbb{Z} \\ L & \mapsto & \left(\tilde{I}_{-1} \to Aut\left(L_{-1}\right),\ \tilde{I}_{-\omega} \to Aut\left(L_{-\omega}\right)\right) \end{array} \tag{45}$$

with the image of Λ going to the generator of

$$\mathbb{Z}/4\mathbb{Z}\times_{\mathbb{Z}/2\mathbb{Z}}\mathbb{Z}/6\mathbb{Z} = \mathbb{Z}/12\mathbb{Z}.$$

The proof that (45) is injective is that any L in the kernel does give flat descent data for a (necessarily trivial) line bundle on $\mathbb{A}^1$ and so must be trivial itself. □

10. $\overline{M}_{0,n}$ a Deligne-Mumford stack vs. the Artin stack $\mathfrak{M}_{0,n}$

We define $\overline{M}_{0,n}$ to be the Deligne-Mumford stack of flat families of stable n-pointed connected curves of arithmetic genus 0. In fact $\overline{M}_{0,n}$ is a projective variety. If we eliminate the condition of stability, that is, if we remove the lower bound of 3 from the number of marked and crossing points on each component of the curve, then the resulting stack, which we call $\mathfrak{M}_{0,n}$ is no longer Deligne-Mumford but only Artin. To see this, notice that a forgetful map from the disjoint union

$$\bigcup_k \overline{M}_{0,n+k} \to \mathfrak{M}_{0,n}$$

gives a smooth cover. (The domain is of course not of finite type). Notice that the image of any fixed $\overline{M}_{0,n+k}$ is a compact, open dense in $\mathfrak{M}_{0,n}$ so that $\mathfrak{M}_{0,n}$ is highly non-separated. Already for the generic object

$$\zeta \in (\mathfrak{M}_{0,0})_{Spec\mathbb{C}}$$

we have that

$$Isom\,(\zeta,\zeta) = \mathbb{P}GL\,(2)$$

so is not finite, meaning that $\mathfrak{M}_{0,0}$ cannot be a Deligne-Mumford stack. Notice that the "points" $(\mathfrak{M}_{0,0})_{Spec\mathbb{C}}$ consist of all possible rational trees.

Another anomoly of the stacks $\mathfrak{M}_{0,n}$ can already be seen in the case of $\mathfrak{M}_{0,1}$. If

$$\mathfrak{C} \to \mathfrak{M}_{0,0}$$

denotes the universal curve, we have a natural open inclusion

$$\mathfrak{M}_{0,1} \to \mathfrak{C}$$

reflecting the fact that all points except crossing points can be marked points. On the other hand it will turn out that there is also a morphism

$$\mathfrak{C} \to \mathfrak{M}_{0,1}$$

which is generically the "identity" but "splits out" each crossing point in ζ' into a new component with a third distinguished point (all choices of which are equal under $Isom\,(\zeta',\zeta')$). Notice that the composition of these morphisms is not the identity.

If V is now a smooth projective manifold and β is a possible curve homology class in V, we let

$$\overline{M}_{0,n}\,(V,\beta)$$

be the stack of β-valued stable maps of n-pointed genus-zero curves to V (non-stable components are collapsed). We have a natural morphism of algebraic stacks

$$\overline{M}_{0,n}\,(V,\beta) \to \mathfrak{M}_{0,n}$$

which "forgets about" V. This morphism is (strongly) representable as follows. For a scheme S and a morphism

$$\underline{S} \to \mathfrak{M}_{0,n}$$

given by a family of curves C/S, we have the fibered square

$$\begin{array}{ccc} Mor_S\,(C, V\times S) & \to & \underline{S} \\ \downarrow & & \downarrow \\ \overline{M}_{0,n}\,(V,\beta) & \to & \mathfrak{M}_{0,n} \end{array} \qquad (46)$$

where $Mor_S\,(C, V\times S)$ is just the relative Hilbert scheme.

11. Deformation to the normal cone

11.1. **Normal cones.** Deformation to the normal cone, as used by Fulton and MacPherson, plays an important role in their intersection theory of schemes, leading to:

(i) Refined pull-backs of Chow groups under regular embeddings.
(ii) A construction of the intersection ring of a smooth scheme.
(iii) A fast proof of Grothendieck-Riemann-Roch.

Deformation to the normal cone is also constructed in general enough terms to apply to Artin stacks (Kresch's thesis), leading to the very important construction of an intrinsic virtual class due to Behrend and Fantechi.

Let X be a scheme (of finite type over a field k). A sheaf of graded commutative $\mathcal{O}_X$-algebras with 1:

$$\mathcal{A}^\bullet = \mathcal{A}^0 \oplus \mathcal{A}^1 \oplus ...$$

is *normally generated* if it satisfies the following properties:
(a) The canonical map $\mathcal{O}_X \to \mathcal{A}^0$ is surjective,
(b) $\mathcal{A}^1$ is coherent,
(c) $\mathcal{A}^d$ is generated by $\mathcal{A}^1$ for all d.

Example 11.1. Every normally generated sheaf of $\mathcal{O}_X$-algebras is locally of the form $A[x_1, \dots, x_n]/I$ (I a homogeneous ideal) over $\mathrm{Spec}(A)$.

Example 11.2. If $X \subset Y$ is a closed subscheme, then there are two normally generated sheaves of $\mathcal{O}_Y$-algebras:

$$\mathcal{I}^\bullet := \mathcal{O}_Y \oplus \mathcal{I} \oplus \mathcal{I}^2 \oplus \dots$$

and

$$\mathcal{I}^\bullet/\mathcal{I}^{\bullet+1} := \mathcal{O}_Y/\mathcal{I} \oplus \mathcal{I}/\mathcal{I}^2 \oplus \dots$$

where $\mathcal{I}$ is the ideal sheaf of X in Y. The second sheaf is a normally generated sheaf of $\mathcal{O}_X$-algebras, as well.

Example 11.3. If E is a vector bundle on X, then:

$$S^\bullet(E) := \mathcal{O}_X \oplus E \oplus S^2(E) \oplus \dots$$

is a normally generated sheaf of $\mathcal{O}_X$-algebras.

Given a normally generated sheaf $\mathcal{A}^\bullet$ of $\mathcal{O}_X$-algebras, we define:
(i) $C := C(\mathcal{A}^\bullet) := \mathit{Spec}(\mathcal{A}^\bullet)$ is the associated *affine cone* over X,
(ii) $\mathbb{P}(C) := \mathit{Spec}(\mathcal{A}^\bullet)$ is the associated *projective family* over X, and
(iii) $\mathbb{P}(C \oplus 1) := \mathit{Spec}(\mathcal{A}[x]^\bullet)$ is the *projective completion* of C, where

$$\mathcal{A}[x]^\bullet := \mathcal{A}^0 \oplus \dots \oplus (\mathcal{A}^0 x^d \oplus \mathcal{A}^1 x^{d-1} \dots \oplus \mathcal{A}^d) \oplus \dots .$$

These families come equipped with morphisms π, p, q respectively, to X, together with a commuting diagram:

$$\begin{array}{ccccc} \mathbb{P}(C) & \xrightarrow{i} & \mathbb{P}(C \oplus 1) & \xleftarrow{j} & C \\ & \searrow^{p} & \downarrow^{\pi} & \swarrow_{q} & \\ & & X & & \end{array}$$

where i and j are complementary closed and open immersions, respectively.

There is a line bundle $\mathcal{O}_{P(C)}(1)$ on $P(C)$ and surjective map

$$p^*(\mathcal{A}^1) \to \mathcal{O}_{\mathbb{P}(C)}(1)$$

and in the case of $P(C \oplus 1)$, the natural section:

$$s : \mathcal{O}_{\mathbb{P}(C \oplus 1)} \to q^* \mathcal{A}^0 \hookrightarrow q^*(\mathcal{A}[x]^1) = q^*(\mathcal{A}^0 x \oplus \mathcal{A}^1) \to \mathcal{O}_{\mathbb{P}(C \oplus 1)}(1)$$

satisfies $Z(s) = \mathbb{P}(C) \subset \mathbb{P}(C \oplus 1)$. So $\mathbb{P}(C)$ is an effective Cartier divisor in $\mathbb{P}(C \oplus 1)$ (at least when the latter is a variety). More generally, it is always a regular embedding of codimension one.

None of this is surprising if you think about it locally. In that case, suppose $\mathcal{A}^\bullet$ comes from $A[x_1, ..., x_n]/I$ over $X = Spec(A)$. Then:
i) $C = Spec(A[x_1, ..., x_n]/I) \subset \mathbb{A}^n_A$ is an affine scheme over $Spec(A)$.
ii) $\mathbb{P}(C) \subset \mathbb{P}^{n-1}_A$ is the projective scheme defined by I, and
iii) $\mathbb{P}(C \oplus 1) \subset \mathbb{P}^n_A$ is defined by $\langle I \rangle$ (think of the extra variable as x_0).

The diagram

$$\begin{array}{ccc} \mathbb{P}(C) & \rightarrow & \mathbb{P}(1 \oplus C) \\ & & \uparrow \\ & & C \end{array}$$

just locates $\mathbb{P}(C)$ and C as the loci $x_0 = 0$ and $x_0 \neq 0$, respectively(!) No big mystery here! One just has to check that the gluing produces the advertised morphisms and line bundles.

In case $\mathcal{A}^\bullet = \mathcal{I}^\bullet/\mathcal{I}^{\bullet+1}$ (associated to a closed subscheme $X \subset Y$), we denote the associated cone by $C_{X/Y}$, and call it the *normal cone* of X in Y.

11.2. **Geometric Realization of the Normal Cone.** i) If $X \subset Y$ has positive codimension in every component, then $\mathbb{P}(C_{X/Y})$ is the exceptional divisor in the blow-up $\tilde{Y}$ of Y along X.

ii) $\mathbb{P}(C_{X/Y} \oplus 1)$ is always the exceptional divisor in the blow-up $\widetilde{Y \times \mathbb{P}^1}$ of $Y \times \mathbb{P}^1$ along the closed subscheme $X \times \{\infty\}$.

Let

$$\epsilon : \widetilde{Y \times \mathbb{P}^1} \to Y \times \mathbb{P}^1$$

be the blow-down. Then $\epsilon^{-1}(Y \times \{\infty\})$ consists of two components. One is the exceptional divisor $\mathbb{P}(C_{X/Y} \oplus 1)$, and the other is a copy of $\widetilde{X}$, which meets $\mathbb{P}(C_{X/Y} \oplus 1)$ along $\mathbb{P}(C_{X/Y})$. The intersection is also Cartier in $\widetilde{Y}$, as it is the exceptional divisor for that blow-up.

iii) We have the equality

$$C_{X/Y} = \epsilon^{-1}(Y \times \{\infty\}) - \widetilde{Y}.$$

11.3. **Elementary Properties of Cones.** If C, D are the affine cones associated to normally generated sheaves $\mathcal{A}^\bullet$ and $\mathcal{B}^\bullet$, respectively, then

(a) If there is a surjective (graded) homomorphism

$$\mathcal{A}^\bullet \to \mathcal{B}^\bullet$$

of $\mathcal{O}_X$-algebras, there are associated closed immersions

$$D \hookrightarrow C, \mathbb{P}(D) \hookrightarrow \mathbb{P}(C)$$

and

$$\mathbb{P}(D \oplus 1) \hookrightarrow \mathbb{P}(C \oplus 1)$$

compatible with $\mathcal{O}(1)$ and immersions i and j.

(b) The irreducible components C_i of C are affine cones associated to quotients of $\mathcal{A}^\bullet$, corresponding locally to the quotients by prime ideals.

(c) The fiber product

$$C \times_X D$$

is the affine cone associated to the tensor-product $\mathcal{O}_X$ -algebra

$$\mathcal{A}^\bullet \otimes_{\mathcal{O}_X} \mathcal{B}^\bullet$$

(with appropriate grading). In particular, in case $\mathcal{B}^\bullet = S^\bullet(E^*)$ for a vector bundle E on X, we let:

$$C \oplus E := C \times_X E$$

and note that we have already seen a special case of this:

$$C \oplus 1 = C \oplus \mathcal{O}_X = C \times \mathbb{A}^1.$$

(d) If

$$f : X \to Y$$

is a morphism, then the affine cone C' for the $\mathcal{O}_X$-algebra $f^*\mathcal{A}^\bullet$ is the fiber product $C \times_Y X$, and likewise for $P(C)$ and $P(C \oplus 1)$.

11.4. **A Fundamental Example.** Given a closed immersion $i : X \to Y$ and an arbitrary morphism $f : V \to Y$, form the fiber product:

$$\begin{array}{ccc} W & \hookrightarrow & V \\ \downarrow & & f\downarrow \\ X & \overset{i}{\hookrightarrow} & Y \end{array}$$

Let $\mathcal{I}$ and $\mathcal{J}$ be the ideals sheaves for $X \subset Y$ and $W \subset V$, respectively. Then from the surjective homomorphism:

$$f^*\mathcal{I} \to \mathcal{J}$$

we obtain quotients $f^*\mathcal{I}^\bullet \to \mathcal{J}^\bullet$ and $f^*(\mathcal{I}^\bullet/\mathcal{I}^{\bullet+1}) \to \mathcal{J}^\bullet/\mathcal{J}^{\bullet+1}$, hence in particular a closed immersion: $C_{W/V} \hookrightarrow C_{X/Y} \times_Y V$ of schemes over V. If, in addition, $f : V \to Y$ is a closed immersion, then we have a further closed immersion:

$$C_{W/V} \hookrightarrow C_{X/Y} \times_Y V \hookrightarrow C_{X/Y}.$$

The following theorem is now easy, but crucial!

Theorem 11.1. *(Deformation to the Normal Cone) Given $X \subset Y$ a closed immersion, define a cycle map:*

$$\sigma : Z_k(Y) \to Z_k(C_{X/Y}); \quad [V] \mapsto [C_{W/V}]$$

as above, extended by linearity. Then $\sigma(\alpha) \sim 0$ if $\alpha \sim 0$, i.e. σ defines:

$$\overline{\sigma} : A_k(Y) \to A_k(C_{X/Y})$$

Example 11.4. If $\pi : E \to X$ is a vector bundle and $z : X \to E$ is the zero section, then $C_{X/E} = E$ and $\overline{\sigma} = (ident.) : A_k(E) \to A_k(E)$, which we may see as follows. Since the flat pull-back

$$\pi^* : A_{k-e}(X) \to A_k(E)$$

is an isomorphism in this case, we need only compute $\overline{\sigma}$ on varieties of the form $V = \pi^{-1}(W)$. But on such varieties, $C_{W/V} = E|_W = V$.

It is more interesting to compute the inverse of the flat pull-back π^*, which we denote by

$$z^* : A_k(E) \to A_{k-e}(X).$$

This is given by

$$z^*[V] = q_*(c_e(Q) \cap [P(C_{V\cap X/V} \oplus 1)])$$

where Q is the universal quotient bundle on $P(E \oplus 1)$. It is crucial to observe that for fixed V, this map factors through

$$A_{k-e}(V \cap X) \to A_{k-e}(X).$$

A closed immersion $i : X \to Y$ of schemes of finite type over a field is a *regular embedding of codimension* d if it is locally defined by a regular sequence of length d, or equivalently if

$$C_{X/Y} = N_{X/Y}$$

the normal bundle to X in Y. In this case, we use the deformation to the normal cone to define the *Gysin map for regular embeddings*:

$$i^* = z^*\overline{\sigma} : A_k(Y) \to A_{k-d}(X)$$

where $z : Y \to N = C_{X/Y}$ is the zero section of the normal bundle.

The Gysin map has various nice properties, including:

- If X intersects V transversally, then $i^*([V]) = [V \cap X]$.
- If V is contained in X, then $i^*([V]) = c_d(N_{X/Y}) \cap [V]$.

and in general, $i^*[V]$ factors through

$$A_{k-d}(V \cap X) \to A_{k-d}(X).$$

This notion is exploited in the definition of a *refined Gysin map*

$$i^! : A_k(Y') \to A_{k-d}(X')$$

for any fiber square:

$$\begin{array}{ccc} X' & \overset{i'}{\hookrightarrow} & Y' \\ g\downarrow & & f\downarrow \\ X & \overset{i}{\hookrightarrow} & Y \end{array}$$

which opens the door to many functorial properties, for instance:

(a) Refined Gysin maps commute with proper push-forward.
(b) Refined Gysin maps commute with flat pull-back.
(c) Refined Gysin maps commute with capping with Chern classes.
(d) Refined Gysin map commute with each other.
(e) A composition of refined Gysin maps is a refined Gysin map.
(f) Excess intersection. If i' is a regular embedding (of codim $d' \leq d$), then $N' = N_{X'/Y'} \subset g^*N = g^*N_{X/Y}$ is a vector bundle inclusion and

$$i^!(\alpha) = c_{d-d'}(g^*N/N') \cap (i')^* (\alpha)$$

(and similarly for $i'^!$ in the presence of a further map $Y'' \to Y'$)

If X and Y are schemes, we may define the *exterior product*:

$$Z_k(X) \otimes Z_l(Y) \overset{\times}{\to} Z_{k+l}(X \times Y)$$

by setting $[V] \times [W] = [V \times W]$. This is defined on the level of Chow groups.

Now, suppose $f : X \to Y$ is an arbitrary morphism to a nonsingular variety Y of dimension n. Then the graph of f gives

$$\gamma_f : X \to X \times Y,$$

a regular embedding of codimension n, and via the Gysin map and the exterior product, we may define:

$$x \cdot_f y := \gamma_f^*(x \times y)$$

and, in particular,

$$f^*(y) := \gamma_f^*([X] \times y)$$

In case f is the identity map, we get the *intersection pairing*

$$x \cdot y$$

and one checks that $[V] \cdot [W]$ factors through $A_*(V \cap W)$.

11.5. **Intersection Theorem.** The functorial properties of the refined Gysin maps lead to the following:

Theorem 11.2. *Let Y be a nonsingular variety of dimension n.*

(a) Via the intersection pairing, $A^*(Y) = A_{n-*}(Y)$ is a commutative ring with $[Y] = 1$, and the pull-back f^* is a graded ring homomorphism.

(b) (Projection Formula) If $f : X \to Y$ is a proper morphism of non-singular varieties, then

$$f_*(x \cdot f^* y) = f_*(x) \cdot y$$

for all $x \in A^*(X)$ and $y \in A^*(Y)$.

12. Cone stacks

12.1. **Intrinsic normal cone.** Recal that a cone C over a scheme X is an affine surjective morphism

$$f : C \to X$$

such that

$$f_* \mathcal{O}_C := \mathcal{A}_0 \oplus \mathcal{A}_1 \oplus \dots$$

where

$$\mathcal{O}_X \to \mathcal{A}_0$$

is surjective (always an isomorphism in our applications) and $\mathcal{A}_1$ is a coherent $\mathcal{O}_X$-module with the induced map

$$Sym^d \mathcal{A}_1 \to \mathcal{A}_d$$

surjective for each $d \geq 0$. Cones of the form

$$E = Spec\,(Sym \mathcal{A}_1)$$

will be called *abelian cones*. We have a natural *abelianization functor*

$$C \mapsto A\,(C)$$

and a natural imbedding

$$C \hookrightarrow A\,(C)\,.$$

An abelian cone is always an abelian group scheme with addition

$$E \times_X E \to E$$

given by the coaddition

$$\mathcal{A}_1 \to \mathcal{A}_1 \otimes_{\mathcal{O}_X} \quad \mathcal{A}_1$$

given by the diagonal map.

Example 12.1. Suppose we have a closed immersion of schemes

$$i : X \hookrightarrow Y$$

where $\mathcal{I}$ is the ideal sheaf defining X in Y so that we have

$$C_{X \backslash Y} = Spec\left(\bigoplus\nolimits_{d \geq 0} \left(\mathcal{I}^d / \mathcal{I}^{d+1}\right)\right)$$
$$N_{X \backslash Y} = Spec\left(\bigoplus\nolimits_{d \geq 0} Sym^d \left(\mathcal{I}/\mathcal{I}^2\right)\right).$$

If we happen to have a situation in which a map of abelian cones

$$E \to A\left(C\right)$$

factors through C, then E acts on C we can form a stack

$$[C/E]$$

as discussed in previous chapters. Recall from Example 1.8 that if we have a scheme X with a group action by a group G we associated a stack

$$[X/G]$$

and functor

$$\underline{X} \to [X/G].$$

In this instance the group is E and the scheme is C. Thus we have

$$\underline{C} \to [C/E]$$

The cover

$$C' = E \times_X C \to C$$

is a smooth surjective (affine) map so this stack is an *Artin stack* . In fact it is a groupoid scheme.

Example 12.2. Suppose that, for the closed immersion

$$i : X \hookrightarrow Y$$

in the previous example we add the additional condition that Y be smooth. Then letting $\mathfrak{J}$ denote the ideal of the diagonal

$$\Delta \subseteq Y \times Y$$

we can pull $\mathcal{I} = \mathcal{I}_X$ back to Δ via one of the projections

$$\pi : \Delta \to Y$$

and we have induced

$$\mathcal{I}^d \to \mathfrak{J}^d\big|_{\Delta}$$

and so a natural map

$$\mathcal{I}^d / \mathcal{I}^{d+1} \to \mathfrak{J}^d / \mathfrak{J}^{d+1}\big|_{\pi^{-1}(X)} = Sym^d\, \Omega_Y|_X\,.$$

Thus we have

$$E := T_Y \to C.$$

The commutativity of

$$\begin{array}{ccc} \mathcal{I} & \to & \mathcal{I} \otimes \mathfrak{J}|_\Delta \\ \downarrow & & \downarrow \\ \mathfrak{J}|_\Delta & & \mathfrak{J}|_\Delta \otimes \mathfrak{J}|_\Delta \end{array}$$

gives the T_Y-action on $C_{X \setminus Y}$.

The central result we are after is the fact that the normal cone-stack

$$\left[C_{X\setminus Y} / \, T_Y|_X\right]$$

is intrinsic to X, that is, this stack (as a stack over X) does not depend on the smooth scheme Y into which we choose to embed X.

Example 12.3. Let

$$X \subseteq \mathbb{A}^n$$

be a hypersurface defined by a single polynomial f. We then map

$$\overline{f} = \frac{f + \{f^2\}}{\{f^2\}} \mapsto df \in \Omega^1_{\mathbb{A}^n}\big|_X$$

and for the action

$$T_Y|_X \times C \to C$$

we map

$$\overline{f} \mapsto df \otimes 1 + 1 \otimes \overline{f} \in \left(\Omega^1_{\mathbb{A}^n}\big|_X \otimes 1\right) \oplus \left(1 \otimes \mathcal{I}/\mathcal{I}^2\right).$$

So at points where $df = 0$ the action is trivial.

Proposition 12.1. *i)*

$$[C/E]$$

is the Artin stack associated to the groupoid scheme

$$E \times_X C \rightrightarrows C$$

where

$$\begin{array}{rcl} s & = & \textit{action} \\ t & = & \textit{projection.} \end{array}$$

ii) The category

$$Obj\, [C/E]_{f:S\to X}$$

is the set of pairs consisting in a principal E/X-bundle E'/S together with an E/X-equivariant map

$$E'/S \to C.$$

Further we have a **zero-section**

$$\underline{0} : X \to C$$

and an $\mathbb{A}^1$**-action** (rescaling the cone) given by

$$\mathbb{A}^1 \times C \to C$$

given by

$$\begin{array}{rcl} \mathcal{I}^d/\mathcal{I}^{d+1} & \to & k[x] \otimes \mathcal{I}^d/\mathcal{I}^{d+1}. \\ f_1 \cdot \ldots \cdot f_d & \mapsto & x^d \otimes f_1 \cdot \ldots \cdot f_d \end{array}$$

Also we have from Example1.8 we have

$$\begin{array}{ccccc} \underline{E} & \to & \underline{E \times_X C} & \to & \underline{C} \\ \downarrow & & \downarrow & & \downarrow \\ \underline{X} & \to & \underline{C} & \to & [C/E] \end{array}.$$

We shall use $E \times_X C$ as the locally trivial smooth (one-set) affine cover of C in order to do descent arguments. Also we will need the following *Cone Lemma*:

Lemma 12.2. *i) Any commuting diagram of actions*

$$\begin{array}{ccc} E & \xrightarrow{\varphi} & E' \\ \downarrow d & & \downarrow d' \\ C & \xrightarrow{\psi} & C' \end{array}$$

induces a morphism of stacks

$$\psi/\varphi : [C/E] \to [C'/E'] .$$

ii) Given a cone morphism

$$k : C \to E'$$

and defining

$$\begin{array}{rcl} \overline{\varphi} & = & \varphi + k \circ d \\ \overline{\psi} & = & \psi + d' \circ k, \end{array}$$

then

$$\begin{array}{ccc} E & \xrightarrow{\overline{\varphi}} & E' \\ \downarrow d & & \downarrow d' \\ C & \xrightarrow{\overline{\psi}} & C' \end{array}$$

gives a natural isomorphism of functors

$$\psi/\varphi \cong \overline{\psi}/\overline{\varphi}.$$

(Compare with (27)*.)*

iii) If the commutative diagram in i) is cartesian and

$$C \times_X E' \to C'$$

is surjective, then ψ/φ is an isomorphism.

Proof. i) The point is that the diagram

$$\begin{array}{ccc} E \times_X C & \xrightarrow{\varphi} & E' \times_X C' \\ s = d \downdownarrows t = proj. & & s' = d' \downdownarrows t' = proj. \\ C & \xrightarrow{\psi} & C' \end{array}$$

is a morphism of groupoid schemes as in Example 8.1 and, as such, gives a morphism of stacks as in Proposition 8.1 by descent. (The fact that we have a morphism of group actions is compatibility of cocycle conditions necessary for descent.)

ii) In fact Proposition 8.1 gives a functor from the category of groupoid schemes to the category of stacks. This means that a morphism

$$\begin{array}{ccc} R & \rightarrow & R' \\ \downdownarrows & & \downdownarrows \\ U & \rightarrow & U' \end{array}$$

of groupoid schemes gives a morphism

$$[R \rightrightarrows U] \Longrightarrow [R' \rightrightarrows U']$$

of stacks and a two-morphism

$$\begin{array}{ccc} & & \begin{array}{ccc} R & \rightarrow & R' \\ \downdownarrows & & \downdownarrows \\ U & \rightarrow & U' \end{array} \\ & \swarrow \alpha & \\ \begin{array}{ccc} R & \rightarrow & R' \\ \downdownarrows & & \downdownarrows \\ U & \rightarrow & U' \end{array} & & \end{array}$$

(see 27) gives a natural transformation between morphisms of stacks, which in this case looks like

$$\begin{array}{ccc} & & \begin{array}{ccc} E \times_X C & \xrightarrow{(\varphi,\psi)} & E' \times_X C' \\ \downdownarrows & & \downdownarrows \\ C & \xrightarrow{\psi} & C' \end{array} \\ & \swarrow (k, \psi) & \\ \begin{array}{ccc} E \times_X C & \xrightarrow{(\overline{\varphi},\overline{\psi})} & E' \times_X C' \\ \downdownarrows & & \downdownarrows \\ C & \xrightarrow{\overline{\psi}} & C' \end{array} & & \end{array}$$

where

$$\begin{aligned} s' \circ (k, \psi) &= \psi + d' \circ k \\ m\left((k, \psi) \circ s, (\varphi, \psi)\right) &= (k \circ d + \varphi, \psi)\,. \end{aligned}$$

This is a 2-isomorphism since it has an inverse

$$\begin{aligned} \varphi &= \overline{\varphi} - k \circ d \\ \psi &= \overline{\psi} - d' \circ k. \end{aligned}$$

iii) We first reduce to the case in which

$$C \rightarrow C'$$

is surjective by noting that we can replace

$$E \to C$$

with

$$E \times_X E' \to C \times_X E'$$

without losing either of the hypotheses. The assertion then follows by descent. Namely, given a diagram

$$\begin{array}{ccc} E \times_X C & \to & C \\ \downarrow & & \\ C & & \end{array} \tag{47}$$

there is a unique minimal object that can fill in the lower right-hand corner to make a cartesian square, namely

$$[C/E]\,.$$

So, if we can show that the diagram is also minimally cartesian if we fill in with

$$[C'/E']$$

then the two are isomorphic by a unique isomorphism. To see this let "⊡" denote that a diagram is cartesian and consider the following expanded diagram

$$\begin{array}{ccccc} E \times_X C & & \to & & C \\ \downarrow & & \boxdot & & \downarrow \\ E' \times_X C & \to & E' \times_X C' & \to & C' \\ \downarrow & \boxdot & \downarrow & \boxdot & \downarrow \\ C & \to & C' & \to & [C'/E'] \end{array}$$

Notice that we use the surjectivity of

$$C \to C'$$

to get surjectivity of

$$C \to [C'/E']$$

which assures minimality. □

Theorem 12.3. *If Y is smooth, then, up to canonical isomorphism,*

$$\left[C_{X\backslash Y} /\, T_Y|_X\right]$$

depends only on X, that is, is independent of the choice of the smooth scheme Y into which X is embedded.

Proof. We can reduce to the case of comparing

$$\begin{array}{ccc} & & Y \\ & \nearrow_i & \\ X & & \uparrow_p \\ & {}_j\searrow & \\ & & Z \end{array} \tag{48}$$

in the case in which there is a smooth surjective morphism p relating Y and Z via the above commutative diagram since any two embeddings i and j of X can be compared in this way with the embedding into the product space $Y \times Z$. So we assume the situation of (48). Now let

$$T_p$$

be the fiber tangent bundle for p. We apply Lemma 12.2 to the commutative diagram of actions

$$\begin{array}{ccccccccc} 0 & \to & T_p|_X & \to & T_Z|_X & \to & T_Y|_X & \to & 0 \\ & & \updownarrow & & \downarrow & \boxdot & \downarrow & & \\ 0 & \to & T_p|_X & \to & C_{X\setminus Z} & \to & C_{X\setminus Y} & \to & 0 \end{array} \tag{49}$$

by noticing notice that the square of $\boxdot$ is cartesian and its bottom horizontal map is surjective. ☐

13. h^1/h^0-resolution of intrinsic normal cone

Given the data of a morphism

$$\mathfrak{E}^{-1} \to \mathfrak{E}^0$$

from a coherent sheaf to a locally free coherent sheaf, we let

$$\begin{array}{rcl} E_0 & = & Spec\left(Sym\mathfrak{E}^0\right) \\ E_1 & = & Spec\left(Sym\mathfrak{E}^{-1}\right) \end{array}$$

be the corresponding abelian cones. So we have a cone-stack

$$[E_1/E_0] =: \left(h^1/h^0\right)\left(\mathfrak{E}^\bullet\right)$$

given by the action

$$E_0 \to E_1.$$

Lemma 13.1. *Given a morphism*

$$\varphi : \mathfrak{E}^\bullet \to \mathfrak{F}^\bullet$$

of data as above, and corresponding morphism

$$\overline{\varphi} : [F_1/F_0] \to [E_1/E_0]$$

of cone stacks,

i) $\overline{\varphi}$ is a representable morphism of Artin stacks (in the sense of algebraic spaces) if and only, for the maps

$$h^i(\varphi) : H^i\left(\mathfrak{E}^\bullet\right) \to H^i\left(\mathfrak{F}^\bullet\right),$$

$h^0(\varphi)$ is surjective;

ii) if $h^0(\varphi)$ is also injective and $h^{-1}(\varphi)$ is surjective, a representable morphism $\overline{\varphi}$ is a closed immersion,

iii) $\overline{\varphi}$ is an isomorphism if and only if φ is a quasi-isomorphism.

Proof. i) Suppose $h^0(\varphi)$ is surjective. We first show that it suffices to consider only the case

$$\varphi : \mathfrak{E}^\bullet \to \mathfrak{F}^\bullet$$

with

$$\mathfrak{E}^0 = \mathfrak{F}^0.$$

To see this, choose a vector bundle $\mathfrak{G}$ such that we have a surjection

$$\mathfrak{G} \twoheadrightarrow \mathfrak{F}^{-1}$$

and replace $\mathfrak{E}^\bullet$ with

$$\mathfrak{E}^{-1} \oplus \mathfrak{G} \to \mathfrak{E}^0 \oplus \mathfrak{G}$$

so that we have the diagram

$$\begin{array}{ccc} \mathfrak{E}^{-1} \oplus \mathfrak{G} & \to & \mathfrak{E}^0 \oplus \mathfrak{G} \\ \downarrow & & \downarrow \\ \mathfrak{F}^{-1} & \xrightarrow{\overline{\varphi}} & \mathfrak{F}^0 \end{array}$$

with both vertical maps surjective and

$$[E_1/E_0] = [(E_1 \oplus G) / (E_0 \oplus G)]$$

by Lemma 12.2. Let

$$\mathfrak{K} = \ker\left(\mathfrak{F}^{-1} \oplus \mathfrak{E}^0 \oplus \mathfrak{G} \to \mathfrak{F}^0\right).$$

Then we have the cartesian diagram

$$\begin{array}{ccc} \mathfrak{K} & \to & \mathfrak{E}^0 \oplus \mathfrak{G} \\ \downarrow & & \downarrow \\ \mathfrak{F}^{-1} & \to & \mathfrak{F}^0 \end{array}$$

Thus

$$[K/(E_0 \oplus G)] \to [F_1/F_0]$$

is an isomorphism by Lemma 12.2 since, by construction, we have the cartesian diagram

$$\begin{array}{ccc} F_0 & \to & F_1 \\ \downarrow & & \downarrow \\ E_0 \oplus G & \to & K \end{array}$$

with

$$E_0 \oplus G \oplus F_1 \to K$$

surjective. Thus we are reduced to considering only the case

$$\varphi : \mathfrak{E}^\bullet \to \mathfrak{F}^\bullet$$

with

$$\mathfrak{E}^0 = \mathfrak{F}^0.$$

Then $E_0 = F_0$.

What we must show is that, for any algebraic space S and morphism

$$\begin{array}{ccc} & & \underline{S} \\ & & \downarrow \\ [F_1/F_0] & \xrightarrow{\overline{\varphi}} & [E_1/E_0] \end{array}$$

we can find algebraic space W and cartesian diagram

$$\begin{array}{ccc} W \times_X F_0 & \to & S \times_X E_0 \\ \downarrow & & \downarrow \\ F_1 & \to & E_1 \end{array},$$

that is, a cartesian diagram

$$\begin{array}{ccc} \underline{W} & \rightarrow & \underline{S} \\ \downarrow & & \downarrow \\ [F_1/F_0] & \xrightarrow{\overline{\varphi}} & [E_1/E_0] \end{array} .$$

But using

$$\underline{S} \rightarrow [E_1/E_0]$$

we can let the fibered product

$$\begin{array}{ccc} E_0 \times S \times_{E_1} F_1 & \rightarrow & E_0 \times_X S \\ \downarrow & & \downarrow \\ F_1 & \rightarrow & E_1 \end{array}$$

be acted on (freely) by $E_0 = F_0$ to give W the cartesian square

$$\begin{array}{ccc} [(E_0 \times S \times_{E_1} F_1)/E_0] & \rightarrow & [E_0 \times S/E_0] \\ \downarrow & & \downarrow \\ [F_1/F_0] & \rightarrow & [E_1/E_0] \end{array} .$$

Conversely suppose $h^0(\varphi)$ is not surjective. Then neither is the map of vector bundles

$$\mathfrak{E}^0 \rightarrow \mathfrak{F}^0.$$

Let S be a Zariski open subset of the support of the cokernel on which it is locally free. Since everything is natural under pull-back to S, we can assume that $S = X$ and

$$F_0 \rightarrow E_0$$

has non-trivial vector-bundle kernel. In fact by shrinking S we can assume that

$$\begin{array}{ccc} F_0 & \rightarrow & E_0 \\ \downarrow & & \downarrow \\ F_1 & \rightarrow & E_1 \end{array}$$

is a commutative diagram of constant-rank morphisms of vector bundles such that, for

$$\begin{array}{ccc} 0 & \rightarrow & K \rightarrow E_0 \rightarrow E_1 \\ 0 & \rightarrow & L \rightarrow F_0 \rightarrow F_1 \end{array}$$

exact, the induced map

$$L \rightarrow K$$

has non-trivial (vector-bundle) kernel M. Then E_1/E_0 and F_1/F_0 are just vector bundle schemes and the constant-rank morphism of vector bundles

$$F_1/F_0 \rightarrow E_1/E_0.$$

But there is no

$$W \rightarrow X$$

such that, as a morphism of stacks

$$\begin{array}{ccc} W \times_X F_0 & \rightarrow & E_0 \\ \downarrow & & \downarrow \\ F_1 & \rightarrow & E_1 \end{array}$$

is cartesian, since

$$W \times_X F_0 \to F_1 \times_{E_1} E_0$$

always has kernel containing

$$W \times_X M.$$

ii) If $h^0(\varphi)$ is also injective and $h^{-1}(\varphi)$ is surjective, then, as above, we can assume we have a diagram

$$\begin{array}{ccccccc} h^1(\mathfrak{E}^\bullet) & \to & \mathfrak{E}^{-1} & \to & \mathfrak{E}^0 & \to & h^0(\mathfrak{E}^\bullet) \\ \downarrow surj. & & \downarrow \varphi^{-1} & & \updownarrow & & \updownarrow \\ h^1(\mathfrak{F}^\bullet) & \to & \mathfrak{F}^{-1} & \to & \mathfrak{E}^0 & \to & h^0(\mathfrak{F}^\bullet) \end{array} \tag{50}$$

so that, by the Snake Lemma, φ^{-1} is also surjective so that

$$F_1 \to E_1$$

is a closed immersion. So

$$F_1/E_0 \to E_1/E_0$$

is a closed immersion.

iii) In (50) φ is a quasi-isomorphism if and only if it is an isomorphism. □

14. Basic example

Suppose we have the following cartesian diagram of scheme maps

$$\begin{array}{ccc} X & \xrightarrow{j} & V \\ \downarrow g & \boxdot & \downarrow f \\ Y & \xrightarrow{i} & W \end{array}$$

where V and W are smooth and

$$i : Y \to W$$

is a regular embedding, that is, Y is a closed subscheme of W defined by an ideal $\mathcal{J}$ given by a regular sequence (lci). Let

$$\mathfrak{E}^\bullet : g^* N^\vee_{Y \setminus W} \to j^* \Omega^1_V$$

defined by the natural maps

$$g^* N^\vee_{Y \setminus W} \to (i \circ g)^* \Omega^1_W \to j^* \Omega^1_V.$$

Let

$$\mathcal{I} = f^* \mathcal{J}$$

be the ideal defining X in V and put

$$\mathfrak{F}^\bullet = \mathfrak{L}^\bullet$$

where

$$\mathfrak{L}^\bullet : \mathcal{I}/\mathcal{I}^2 \to j^* \Omega^1_V.$$

This gives the morphism of cone stacks

$$\left[C_{X \setminus V}/j^* T_V\right] \to [E_1/E_0]$$

which is a closed immersion since, in the diagram

$$\begin{array}{ccc} g^*N^\vee_{Y\backslash W} & \to & j^*\Omega^1_V \\ \downarrow surj. & & \updownarrow \\ \mathcal{I}/\mathcal{I}^2 & \to & j^*\Omega^1_V \end{array},$$

we have that

$$h^0(\varphi)$$

is an isomorphism by construction and

$$h^{-1}(\varphi)$$

is surjective since φ^{-1} is.

In ths situation, we also have a cartesian diagram

$$z : V \to g^*N_{Y\backslash W}$$

is a regular embedding given by the zero section of a bundle of rank d. So we can use Fulton-MacPherson intersection theory to construct a virtual fundamental class by intersecting the image of

$$C_{X\backslash V}$$

with the zero section to obtain the class

$$z^!\left(C_{X\backslash V}\right) \in A^{\operatorname{co\,dim} C_{X\backslash V}+d}(X)$$

in the Chow ring of X. This class will only depend on the cone stack

$$\left[C_{X\backslash V}/E_0\right]$$

and will be called the *virtual fundamental class.*

15. The case of moduli stacks

Our problem will be to put a virtual fundamental class on

$$M := Mor(C, V) = \{c : C \to V\}$$

where V is a smooth projective manifold and C is a Gorenstein curve. Now M is realized as a closed (therefore proper) subscheme

$$Mor(C, V) \subseteq Hilb_{1,p}(C \times V)$$

of virtual dimension

$$\chi\left(c^*(T_V)\right)$$

where p is some fixed Hilbert polynomial given by choosing a polarization of V. Putting

$$M \times C \stackrel{(\pi,f)}{\longrightarrow} M \times V$$

and letting

$$\omega := \omega_{M\times C/M}$$

denote the pull-back of the dualizing line bundle of C, Behrend and Fantechi define a morphism

$$\begin{array}{c} RHom\left(R\pi_*\left(f^*T_V\right), \mathcal{O}_M\right) = R\pi_*\left(RHom\left(f^*T_V, \omega\right)\right) \\ \downarrow \\ R\mathfrak{L}^\bullet \end{array}$$

where $\mathfrak{L}^{\bullet}$ is the cotangent complex associated to a closed embedding

$$M \hookrightarrow Y$$

with Y smooth. The problem is then to find a Y and a two-term complex of bundles

$$\mathfrak{E}^{\bullet}$$

giving $R\pi_*(RHom(f^*T_V, \omega))$ in the derived category and with a commutative diagram

$$\begin{array}{ccc} \mathfrak{E}^{-1} & \rightarrow & \mathfrak{E}^{0} \\ \downarrow surj. & & \updownarrow = \\ \mathcal{I}/\mathcal{I}^{2} & \rightarrow & \Omega^1_Y\big|_M \end{array} . \tag{51}$$

This proceeds roughly as follows. For example, let

$$M = \overline{M}_{0,0}(V, \beta)$$

denote the Deligne-Mumford moduli stack of stable maps of zero-pointed curves of genus zero into the smooth projective manifold V. We have a diagram

$$\begin{array}{ccc} C & \xrightarrow{e} & V \\ \downarrow \pi & & \\ M & & \end{array} .$$

Let L be an ample line bundle on V and let ω_π denote the relative dualizing sheaf. Stability implies that

$$\omega_\pi \otimes e^* L^3$$

is relatively ample. For $N >> 0$,

$$\pi^*\pi_*\left(e^*\left(T_V \otimes L^N\right)\right) \rightarrow e^*\left(T_V \otimes L^N\right)$$

is surjective. Let K denote the kernel of the surjection

$$e^*L^{-N} \otimes \pi^*\pi_*\left(e^*\left(T_V \otimes L^N\right)\right) \rightarrow T_V.$$

Then applying $R\pi_*$ to the exact sequence

$$0 \rightarrow K \rightarrow e^*L^{-N} \otimes \pi^*\pi_*\left(e^*\left(T_V \otimes L^N\right)\right) \rightarrow T_V \rightarrow 0$$

we obtain $\longrightarrow$

$$0 \rightarrow \pi_*T_V \rightarrow R^1\pi_*K \rightarrow R^1\pi_*\left(e^*L^{-N}\right) \otimes \pi_*\left(e^*\left(T_V \otimes L^N\right)\right) \rightarrow R^1\pi_*T_V \rightarrow 0.$$

Thus we have that, since

$$R^1\pi_*K \rightarrow R^1\pi_*\left(e^*L^{-N}\right) \otimes \pi_*\left(e^*\left(T_V \otimes L^N\right)\right)$$

is a complex of vector bundles

$$RHom\left(R\pi_*\left(f^*T_V\right), \mathcal{O}_M\right)$$

is just the complex of vector bundles

$$\mathfrak{E}^{\bullet} = \pi_*\left(e^*L^N \otimes \omega_\pi\right) \otimes R^1\pi_*\left(e^*\left(T_V \otimes L^{-N}\right) \otimes \omega_\pi\right) \rightarrow \pi_*\left(K^\vee \otimes \omega_\pi\right)$$

Before we can see how to map (51) is constructed, we need to return to the cotangent complex. The basic idea of the cotangent complex is to associate to any morphism

$$f : X \rightarrow Y$$

of schemes a complex

$$\mathfrak{L}^{\bullet}_{X/Y}$$

(in the (bounded) derived category) which plays the role played by the sheaf of relative differentials (in degree 0) in the case in which f is a smooth surjective morphism. The complex $\mathfrak{L}^{\bullet}_{X/Y}$ has, in general, only non-positive terms and

$$\mathcal{H}^0\left(\mathfrak{L}^{\bullet}_{X/Y}\right) = \Omega_{X/Y}.$$

Also, for any sequence of maps

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

we have an exact triangle

$$\begin{array}{ccc} f^*\mathfrak{L}^{\bullet}_{Y/Z} & \rightarrow & \mathfrak{L}^{\bullet}_{X/Y} \\ \nwarrow & & \swarrow \\ & \mathfrak{L}^{\bullet}_{X/Y} & \end{array}$$

derived from the exact sequence

$$f^*\Omega_{Y/Z} \to \Omega_{X/Z} \to \Omega_{X/Y} \to 0.$$

When

$$f : X \to W$$

is a closed embedding with ideal $\mathcal{I}$, we have a mapping of complexes

$$\begin{array}{ccccc} 0 & \to & \mathcal{I}/\mathcal{I}^2 & \to & \Omega_W|_X \\ & & \downarrow & & \downarrow \\ \cdots & \to & \mathfrak{L}^{-1}_{X/W} & \to & \mathfrak{L}^0_{X/W} \end{array}$$

yielding an isomorphism at $\mathcal{H}^0$. If in addition Y is smooth, then

$$\ker\left(\mathcal{I}/\mathcal{I}^2 \to \Omega_W|_X\right) = \mathcal{H}^{-1}\left(\mathfrak{L}^{\bullet}_{X/W}\right).$$

In fact $\mathfrak{L}^{\bullet}_{X/Y}$ is defined for any (strongly) representable morphism of algebraic spaces

$$f : \underline{X} \to \underline{W}.$$

In our case

$$\begin{array}{rcl} \underline{X} & = & \overline{M}_{0,0}(V,\beta) \\ \underline{W} & = & \mathfrak{M}_{0,0} \end{array}$$

and the morphism f is the forgetful morphism given in (46). Somehow f will be a closed embedding in the smooth stack $\mathfrak{M}_{0,0}$ of dimension -3.

16. Intersection theory in stacks

The prototype for a stack is a linear algebraic group G acting on a projective variety X giving a quotient stack

$$\mathfrak{X} = [X/G].$$

Examples are $\overline{\mathfrak{M}}_{g,n}$ and $\overline{\mathfrak{M}}_{g,n}(X,\beta)$. Fundamental consequences of the theory are:
1) Every coherent sheaf on $[X/G]$ is a quotient of a locally free sheaf.
2) There is an intersection theory (Totaro-Edidin-Graham).
The idea is that there is a vector bundle

$$\mathfrak{E} \to \mathfrak{X}$$

with an open sub-algebraic-space $\mathfrak{E}^0 \subseteq \mathfrak{E}$ which is dense in every fiber (in fact the codimension of the complement of $\mathfrak{E}^0$ in $\mathfrak{E}$ can be made large). For example, take a faithful representation

$$G \subseteq GL(W).$$

Then the action of G on $V = W^{\oplus n}$ is free on an open set V^0 with complement of high codimension. Then

$$\mathfrak{E} = \left[X \times W^{\oplus n}/G\right]$$

becomes a vector bundle on $\mathfrak{X}$ and we put

$$\mathfrak{E}^0 = \left[X \times V^0/G\right].$$

Definition 16.1. We define Chow groups

$$\mathfrak{A}^k(\mathfrak{X}) = \mathfrak{A}^k\left(\mathfrak{E}^0\right).$$

These Chow groups have all the usual properties.

Example 16.1.

$$\begin{aligned} G &= GL(n) \\ X &= pt. \\ \mathfrak{X} &= BGL(n) \\ V &= (\mathbb{C}^n)^N \\ V^0 &= \{bases\ of\ \mathbb{C}^n\} \end{aligned}$$

Then

$$\mathfrak{A}^k(\mathfrak{X}) = \frac{\mathbb{Z}[c_1,\ldots,c_n]}{\{relations\ in\ \deg \geq N\}}.$$

Since these relations are in very high codimension we conclude

$$\mathfrak{A}^k(\mathfrak{X}) = \mathbb{Z}[c_1,\ldots,c_n].$$

Lemma 16.1. *If*

$$\pi : \mathfrak{F} \to \mathfrak{X}$$

is a vector bundle, then the natural map

$$\pi^* : \mathfrak{A}^k(\mathfrak{X}) \to \mathfrak{A}^k(\mathfrak{F})$$

is an isomorphism.

Proof. We have the diagram

$$\begin{array}{ccc} F & \to & \mathfrak{F} \\ \downarrow & & \downarrow \\ X & \to & \mathfrak{X} \end{array}$$

where F is a G-equivariant vector bundle. Choose V and V^0 as before. Then we still have a quotient vector bundle

$$\begin{array}{ccc} F \times V^0 & \to & F \times V^0/G \\ \downarrow & & \downarrow \\ X \times V^0 & \to & X \times V^0/G \end{array}$$

so pull back is still an isomorphism of Chow groups since Chow groups descend under a free action. □

16.1. **Defining** h^1/h^0**.** Suppose X is a Deligne-Mumford stack and we have a morphism

$$F^0 \to F^1$$

of vector bundles over X.

Definition 16.2.

$$\left(h^1/h^0\right)[F^*] := \left(\left[F^1/F^0\right] \to X\right).$$

Theorem 16.2. *(Kresch) The pullback map induces an isomorphism*

$$\mathfrak{A}^*(X) \to \mathfrak{A}^*\left(\left[F^1/F^0\right]\right).$$

Proof. (Idea) For the diagram

$$\begin{array}{ccc} F^1 & \xrightarrow{\sigma} & \left[F^1/F^0\right] \\ \downarrow^{\rho} & & \downarrow \\ X & = & X \end{array}$$

ρ^* is an isomorphism of Chow groups. One must show as above that σ^* is too. □

Now suppose that

$$L_X^*$$

is the cotangent complex. L_X^* is locally of the form

$$\ldots \to \frac{\mathcal{I}}{\mathcal{I}^2} \to \Omega_M|_X \to 0$$

where $X \to M$ is a closed embedding and $\mathcal{I}$ is the ideal sheaf of X in M.

Definition 16.3. A (perfect) obstruction theory is a diagram

$$\begin{array}{ccc} E^{-1} & \to & E^0 \\ \downarrow \alpha^{-1} & & \downarrow \alpha^0 \\ \frac{\mathcal{I}}{\mathcal{I}^2} & \to & \Omega_M|_X \end{array}$$

such that the E^i are vector bundles on X and α^0 is an isomorphism and α^{-1} is surjective.

Given a perfect obstruction theory, apply $Hom\,(\ ,\mathcal{O}_X)$ to get

$$\begin{array}{ccc} F^0 & \to & F^1 \\ \uparrow & & \uparrow \\ T_M|_X & \to & N_{X\backslash M} \end{array}.$$

In fact, taking *Spec* of the induced maps

$$\begin{array}{ccc} S^*E^{-1} & \to & S^*E^0 \\ \downarrow \alpha^{-1} & & \downarrow \alpha^0 \\ \sum_{r\geq 0} \frac{\mathcal{I}^r}{\mathcal{I}^{r+1}} & \to & S^*\,\Omega_M|_X \end{array}$$

we obtain

$$\begin{array}{ccc} F^0 & \to & F^1 \\ \uparrow \alpha_0 & & \uparrow \alpha_1 \\ T_M|_X & \to & C_X M \end{array}$$

with α_0 an isomorphism and α_1 injective. This diagram gives

$$\mathfrak{C}_X = [C_X M/\ T_M|_X] \subseteq h^1/h^0\,(F^*)\,.$$

Now

$$\dim \mathfrak{C}_X = 0$$

since

$$\dim T_M|_X = \dim C_X M.$$

Lemma 16.3. *If*

$$F^0 \to F^1$$

is a morphism of vector bundles over a space X, then

$$\pi^* : \mathfrak{A}^*\,(X) \to \mathfrak{A}^*\left(\left[F^1/F^0\right]\right)$$

is an isomorphism.

Proof. Consider the diagram

$$\begin{array}{ccc} F^0 & \to & F^0 \times_X F^1 \\ \downarrow = & & \downarrow \\ F^0 & \to & F^1 \end{array}.$$

Then

$$F^1 = \left[F^0 \times_X F^1/F^0\right]$$

is a vector bundle over

$$\left[F^1/F^0\right].$$

So

$$\mathfrak{A}^*\left(\left[F^1/F^0\right]\right) \cong \mathfrak{A}^*\left(\left[F^0 \times_X F^1/F^0\right]\right) \cong \mathfrak{A}^*\left(F^1\right) \cong \mathfrak{A}^*\,(X)\,.$$

□

Definition 16.4. The virtual fundamental class of E^* is the unique class $\alpha \in \mathfrak{A}^*\,(X)$ such that $\pi^*\,(\alpha) = [C_X M/\ T_M|_X]$.

So

$$\begin{aligned}\dim(\alpha) &= \dim \mathfrak{C}_X - \left(rankF^1 - rankF^0\right) \\ &= rankE^0 - rankE^{-1}.\end{aligned}$$

Example 16.2. Suppose X is smooth. Then we have the diagram

$$\begin{array}{ccc} E^{-1} & \to & E^0 \\ \downarrow & & \downarrow \\ 0_X & \to & \Omega_X \end{array}$$

so that

$$\mathfrak{C}_X = [0_X/T_X]$$

and

$$\begin{array}{ccc} F^0 & \to & F^1 \\ \uparrow = & & \uparrow \\ T_X & \to & 0_X \end{array}$$

so that

$$cokernel\left(F^0 \to F^1\right) = F^1$$

and

$$\alpha = c_{top}F^1.$$

Example 16.3. Suppose X is defined by a cartesian diagram

$$\begin{array}{ccc} X & \hookrightarrow & Y \\ \downarrow^f & & \downarrow^g \\ P & \hookrightarrow & M \end{array}$$

where P, M, and Y are all smooth. Let $\mathcal{J}$ denote the ideal of P in M and $\mathcal{I}$ be the pullback ideal defining X in Y. Then we have

$$\begin{array}{ccc} f^*\frac{\mathcal{J}}{\mathcal{J}^2} = E^{-1} & \stackrel{a}{\to} & \Omega_Y|_X = E^0 \\ \downarrow \alpha^{-1} & & \downarrow \alpha^0 \\ \frac{\mathcal{I}}{\mathcal{I}^2} & \to & \Omega_Y|_X \end{array}$$

where a is the composition

$$f^*\frac{\mathcal{J}}{\mathcal{J}^2} \to f^*\ \Omega_M|_P \to \Omega_Y|_X\,.$$

This gives

$$\begin{array}{ccc} T_Y|_X & \stackrel{a^*}{\to} & f^*N_{P\backslash M} \\ \uparrow \alpha_0 & & \uparrow \alpha_1 \\ T_Y|_X & \to & C_XY \end{array}$$

where the top map factors as

$$T_Y|_X \to f^*\ T_M|_P \to f^*N_{P\backslash M}.$$

So

$$\mathfrak{C}_X = [C_XY/\ T_Y|_X] \subseteq h^1/h^0\left(T_Y|_X \to f^*N_{P\backslash M}\right).$$