Notes on birth, death and immigration processes.

Death process: Suppose individuals die at per capita rate δ.

A deterministic differential equation is $\frac{dN}{dt} = -\delta N$ with solution $N(t) = N(0)\,e^{-\delta t}$

But individuals are discrete, so N, the population size, should be an integer.

Approach 1: The binomial distribution

let $p_k(t) = Pr(N(t) = k) = b(k; e^{-\delta t}, N(0))$ because $e^{-\delta t} = S(t)$ is the survivorship probability at time t.

$E(N(t)) = N(0)\,e^{-\delta t}$, matching the deterministic equation.

Approach 2: Differential equations for the probabilities

population size $\quad 0 \xleftarrow{\delta} 1 \xleftarrow{2\delta} 2 \xleftarrow{3\delta} 3 \xleftarrow{4\delta} 4$

$$\frac{dp_0}{dt} = \delta p_1$$
$$\frac{dp_1}{dt} = -\delta p_1 + 2\delta p_2$$
$$\frac{dp_2}{dt} = -2\delta p_2 + 3\delta p_3$$
$$\vdots$$

These can be solved with a clever trick.

The miracle is that a stochastic model of discrete values is described by a deterministic model of continuous probabilities

Immigration process $\quad 0 \xrightarrow{\lambda} 1 \xrightarrow{\lambda} 2 \xrightarrow{\lambda} 3 \xrightarrow{\lambda} 4 \xrightarrow{\lambda}$

$$\frac{dp_0}{dt} = -\lambda p_0$$
$$\frac{dp_1}{dt} = \lambda p_0 - \lambda p_1$$
$$\frac{dp_2}{dt} = \lambda p_1 - \lambda p_2$$
$$\vdots$$

This is the Poisson process with rate λ, so we know

$$p_k(t) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

Birth process $\quad 1 \xrightarrow{\beta} 2 \xrightarrow{2\beta} 3 \xrightarrow{3\beta} 4 \xrightarrow{4\beta} 5 \xrightarrow{5\beta}$

$$\frac{dp_1}{dt} = -\beta p_1$$
$$\frac{dp_2}{dt} = \beta p_1 - 2\beta p_2$$
$$\frac{dp_3}{dt} = 2\beta p_2 - 3\beta p_3$$

Can solve (a bit tricky) to find

$$p_k(t) = e^{-\beta t}(1 - e^{-\beta t})^{k-1}$$ if we start with one individual

which is the geometric distribution.

Then $E(N) = \frac{1}{e^{-\beta t}} = e^{\beta t}$ where $e^{-\beta t}$ acts as the probability of a success in the ~~binomial~~ geometric