

# Notes on birth, death and immigration processes

Death process: Suppose individuals die at per capita rate  $\delta$ .

A deterministic differential equation is  $\frac{dN}{dt} = -\delta N$  with solution  $N(t) = N(0) e^{-\delta t}$

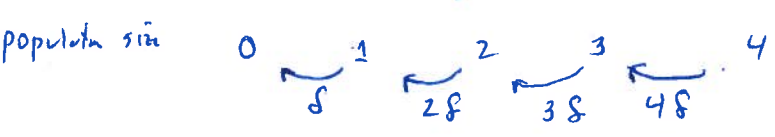
But individuals are discrete, so  $N$ , the population size, should be an integer.

Approach 1: The binomial distribution

Let  $P_N(t) = \Pr(N(t) = k) = b(k; e^{-\delta t}, N(0))$  because  $e^{-\delta t} = S(t)$  is the survivorship probability at time  $t$ .

$E(N(t)) = N(0) e^{-\delta t}$ , matching the deterministic equation.

Approach 2: Differential equations for the probabilities



$$\begin{aligned} \frac{dp_0}{dt} &= -\delta p_0 \\ \frac{dp_1}{dt} &= -\delta p_1 + 2\delta p_2 \\ \frac{dp_2}{dt} &= -2\delta p_2 + 3\delta p_3 \\ &\vdots \end{aligned}$$

These can be solved with a clever trick.

The miracle is that a stochastic model of discrete values is described by a deterministic model of continuous probabilities

Immigration process

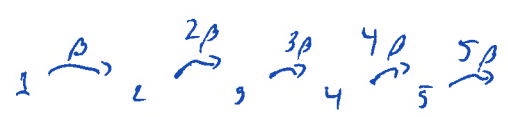


$$\begin{aligned} \frac{dp_0}{dt} &= -\lambda p_0 \\ \frac{dp_1}{dt} &= \lambda p_0 - \lambda p_1 \\ \frac{dp_2}{dt} &= \lambda p_1 - \lambda p_2 \\ &\vdots \end{aligned}$$

This is the Poisson process with rate  $\lambda$ , so we know

$$P_k(t) = \frac{(\lambda t)^k}{k!} e^{-\lambda t}$$

Birth process



$$\begin{aligned} \frac{dp_1}{dt} &= -\beta p_1 \\ \frac{dp_2}{dt} &= \beta p_1 - 2\beta p_2 \\ \frac{dp_3}{dt} &= 2\beta p_2 - 3\beta p_3 \\ &\vdots \end{aligned}$$

Can solve (a bit tricky) to find

$$P_k(t) = e^{-\beta t} (1 - e^{-\beta t})^{k-1} \quad \text{if we start with one individual}$$

Which is the geometric distribution.

The  $E(N) = \frac{1}{e^{-\beta t}} = e^{\beta t}$  where  $e^{-\beta t}$  acts as the probability of a success in the former geometric