Stable Homology by Scanning

Variations on a Theorem of Galatius

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<u>Question</u>: What can one say about $H_*\mathrm{Aut}(F_n)$?

- H_1 and H_2 are known: both are $\mathbb{Z}_2$ for large enough n. (Nielsen, Gersten)
- $H_i\mathrm{Aut}(F_n)$ is finitely generated for all i, n. (Culler–Vogtmann)
- Stability: $\mathrm{Aut}(F_n) \hookrightarrow \mathrm{Aut}(F_{n+1})$ induces $H_i\mathrm{Aut}(F_n) \cong H_i\mathrm{Aut}(F_{n+1})$ for $n > 2i+1$. (H–V 1998)
- $H_i(\mathrm{Aut}(F_n);\mathbb{Q}) = 0$ for $1 \le i \le 6$ except $H_4(\mathrm{Aut}(F_4);\mathbb{Q}) \cong \mathbb{Q}$. (H–V 1998)

<u>Conjecture</u>: $\widetilde{H}_*^{stab}(\mathrm{Aut}(F_n);\mathbb{Q}) = 0$

- $H_*^{stab}\mathrm{Aut}(F_n) \cong H_*^{stab}\Sigma_n \oplus (?)$. (from Waldhausen theory)

<u>Stronger Conjecture</u>: $H_*^{stab}\mathrm{Aut}(F_n) \cong H_*^{stab}\Sigma_n$

Proved by Soren Galatius in 2006.

<u>Remark</u>. $H_*^{stab}(\Sigma_n;\mathbb{Z}_p)$ computed by Nakaoka (1961) — it is large.

Galatius' method: *Scanning*, following Madsen-Weiss (much simplified).

The scanning method shows $H_*^{stab}\mathrm{Aut}(F_n) \cong H_*(\Omega_0^\infty S^\infty)$

where $\Omega_0^\infty S^\infty =$ basepoint component of $\Omega^\infty S^\infty = \cup_n \Omega^n S^n$

Then apply Barrett–Kahn–Priddy Theorem (ca. 1970): $H_*^{stab}\Sigma_n \cong H_*(\Omega_0^\infty S^\infty)$

(Provable by scanning, easy 0-dimensional case.)

The Scanning Method

Given: a sequence of groups $G_1 \subset G_2 \subset \cdots$ that are "automorphism groups" of certain geometric objects.

Goal: compute $H_i^{stab} G_n = H_i(\cup_n G_n)$ (not assuming stability)

3 Main Steps:

1. Construct BG_n as a space of these geometric objects embedded in $\mathbb{R}^\infty = \cup_N \mathbb{R}^N$.
2. Scan to get a spectrum (infinite loopspace) with homology isomorphic to $H_*^{stab}(G_n)$.
3. Identify the spectrum with something nice, preferably known.

The scanning method works for:

- Σ_n and variants, e.g., wreath products $G \wr \Sigma_n$
- MCGs of surfaces and variants, e.g., stabilize both genus and punctures
- $\mathrm{Aut}(F_n)$ and variants, e.g., $\mathrm{Aut}(G * F_n \,\mathrm{rel}\, G)$ for some groups G
- MCGs of 3-dimensional handlebodies and variants
- $\mathrm{Diff}(\#_n S^1 \times S^2)$
- What else?

Also:

- $H_*^{stab} B_n \cong H_*(\Omega_0^2 S^2)$ (F. Cohen)
- $K(\mathbb{Z}, n) =$ free abelian group generated by S^n (Dold–Thom)

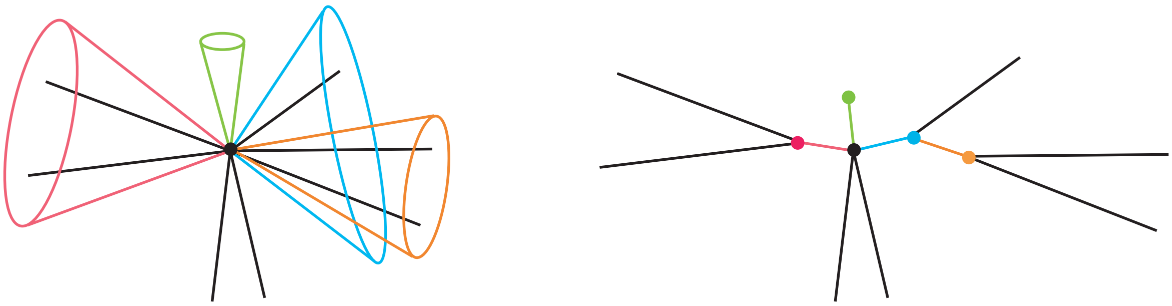
Sketch steps 1–3 for Galatius' theorem

Step 1

Let $\mathcal{G}^N =$ space of finite graphs embedded in $\mathbb{R}^N$:

- smooth edges, linear near vertices
- graphs need not be connected, empty graph allowed
- vertices of valence 0, 1, 2 allowed

Topology on $\mathcal{G}^N$: neighborhood of a given graph consists of graphs obtained by small isotopy and *conical expansion* of vertices to trees:



Disjoint cones, possibly nested, disjoint from the graph except at the vertex.

Translate each cone and everything inside it along the axis of the cone, away from the vertex. The translated vertices of the cones become the new vertices.

(Some differences between $\mathcal{G}^N$ and Galatius' space of graphs.)

Then $\pi_0 \mathcal{G}^N$ = homotopy types of finite graphs for $N \geq 4$.

Let $\mathcal{G}^\infty = \cup_N \mathcal{G}^N$.

Let $\mathcal{G}^\infty_n$ = component of connected rank n graphs: $\pi_1 = F_n$.

<u>Fact 1:</u> $\mathcal{G}^\infty_n \simeq B\text{Out}(F_n)$.

- Main ingredient in proof: contractibility of Outer Space.
- Compare with Igusa theorem: Classifying space of category of rank n finite graphs with morphisms tree collapses is $B\text{Out}(F_n)$.

Similarly: basepointed version of $\mathcal{G}^\infty_n \simeq B\text{Aut}(F_n)$, using graphs with a basepoint vertex at the origin.

<u>Step 2</u>

Enlarge $\mathcal{G}^N$ to a space $\mathcal{G}^{N,N}$ by allowing noncompact graphs extending to infinity in $\mathbb{R}^N$, but properly embedded. In particular, edges can extend to infinity.

Topology: neighborhood of a given graph consists of graphs which are close in a finite ball B, and transverse to ∂B. (Similar to the compact-open topology on function spaces.)

This allows graphs to be pushed to infinity, by pushing radially outward from any point in the complement.

Hence the enlarged space $\mathcal{G}^{N,N}$ is connected.

In fact $\mathcal{G}^{N,N} \simeq S^N$. (This is Step 3.)

Galatius filtration of $\mathcal{G}^{N,N}$:

Let $\mathcal{G}^{N,k}$ = subspace of $\mathcal{G}^{N,N}$ consisting of graphs in $\mathbb{R}^k \times I^{N-k}$, i.e., graphs that can extend to infinity in k directions.

$\mathcal{G}^N \simeq \mathcal{G}^{N,0} \subset \mathcal{G}^{N,1} \subset \cdots \subset \mathcal{G}^{N,N}$

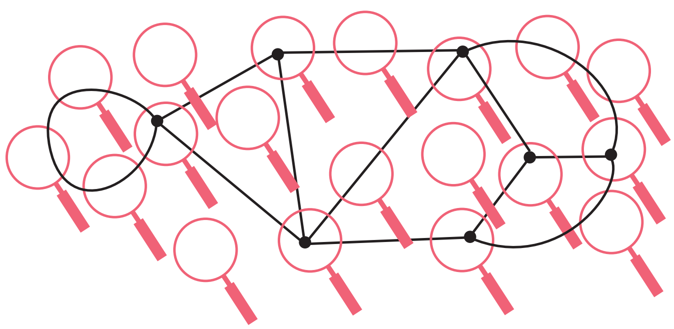
Natural map $\mathcal{G}^{N,k} \to \Omega\mathcal{G}^{N,k+1}$, translate from $-\infty$ to $+\infty$ in $(k+1)$st coordinate. Get a loop based at the empty graph.

Compose:

$\mathcal{G}^{N,0} \to \Omega\mathcal{G}^{N,1} \to \Omega^2\mathcal{G}^{N,2} \to \cdots \to \Omega^N\mathcal{G}^{N,N}$, translate in all directions.

Can rescale graphs in $\mathcal{G}^{N,N}$ to the part lying in a small disk around 0, the germ at 0.

Combined with $\mathcal{G}^{N,0} \to \Omega^N\mathcal{G}^{N,N}$ this is *scanning*: Moving a magnifying lens over all of $\mathbb{R}^N$, scanning an entire graph.



Fact 2.1: $\mathcal{G}^{N,k} \to \Omega\mathcal{G}^{N,k+1}$ is a homotopy equivalence when $k > 0$.

Sketch proof:

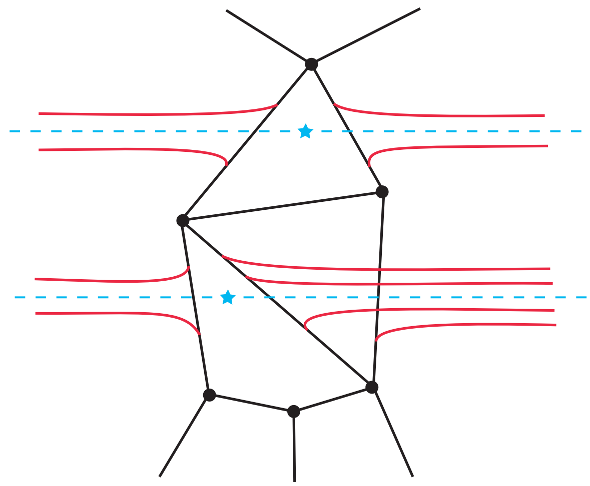
Product in $\mathcal{G}^{N,k}$ when $k < N$: juxtaposition in $(k+1)$st coordinate, then rescale, like composition of loops. Can be improved to be associative — a monoid — like Moore loopspace:

$$\mathcal{M}^{N,k} = \{ (\Gamma, a) \in \mathcal{G}^{N,k} \times [0,\infty) \mid \Gamma \subset \mathbb{R}^k \times [0,a] \times I^{N-k-1} \}$$

Then $\mathcal{M}^{N,k} \simeq \mathcal{G}^{N,k}$.

Claim: $\mathcal{G}^{N,k+1} \simeq B\mathcal{M}^{N,k}$

To prove this, consider the subspace $\mathcal{G}_s^{N,k+1} \subset \mathcal{G}^{N,k+1}$ consisting of split graphs: disjoint from at least one slice $\mathbb{R}^k \times \{a\} \times I^{N-k-1}$. $\mathcal{G}_s^{N,k+1} \simeq \mathcal{G}^{N,k+1}$ if $k > 0$ by pushing radially to ∞ in $\mathbb{R}^k \times \{a\}$ from some point in the complement of a given graph. Do this in many slices independently. Combine by partition of unity.



Furthermore $\mathcal{G}_s^{N,k+1} \simeq B\mathcal{M}^{N,k}$ by sliding the first and last pieces to $\pm\infty$. (Just look at the definition of a classifying space.)

Thus $\mathcal{G}^{N,k+1} \simeq B\mathcal{M}^{N,k}$.

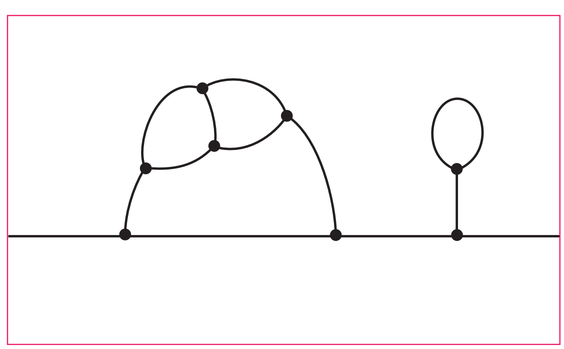
$$
\begin{aligned}
\text{Hence}\quad \Omega\mathcal{G}^{N,k+1} &\simeq \Omega B\mathcal{M}^{N,k} \\
&\simeq \mathcal{M}^{N,k} \qquad \text{since } \pi_0\mathcal{M}^{N,k} = 0 \text{ for } k > 0 \text{ — easy argument.} \\
&\simeq \mathcal{G}^{N,k}
\end{aligned}
$$

□

<u>$k = 0$</u>: Is $\mathcal{G}^{N,0} \to \Omega\mathcal{G}^{N,1}$ a homotopy equivalence? No, since $\pi_0\Omega\mathcal{G}^{N,1}$ is a group but $\pi_0\mathcal{G}^{N,0} \cong \pi_0\mathcal{M}^{N,0}$ is only a monoid. (Other differences: components of $\mathcal{G}^{N,0}$ have different nonabelian π_1's, unlike $\Omega\mathcal{G}^{N,1}$.)

Want to apply the Group Completion Theorem (ca. 1970) to get a stable homology equivalence instead.

New $\mathcal{M}^{N,0}$: monoid $\mathcal{M}^N$ consisting of pairs (Γ, a) with Γ a finite connected graph in $[0,a] \times I^{N-1}$ containing the base line $[0,a] \times \{0\}$. (This follows the later paper of Galatius and Randal–Williams on MCGs, rather than Galatius' original paper.)



Only need the case $N = \infty$.

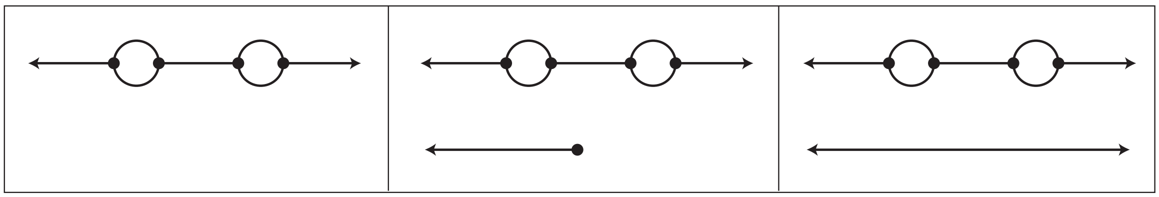
$\mathcal{M}^\infty = \coprod_n \mathcal{M}^\infty_n$ for $\mathcal{M}^\infty_n$ = component with rank n graphs.

$\mathcal{M}^\infty_n \simeq B\mathrm{Aut}(F_n)$ as in Step 1.

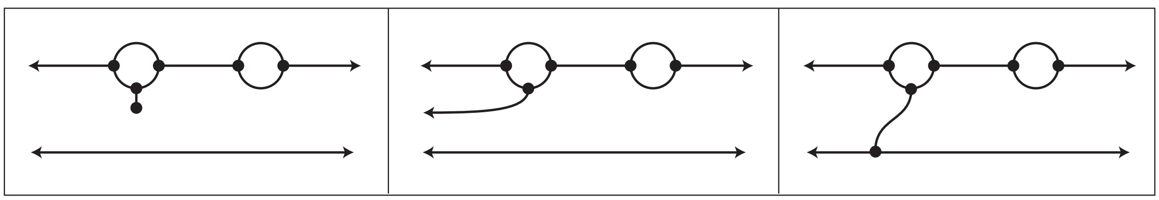
<u>Fact 2.2:</u> $\mathcal{G}^{\infty,1} \simeq B\mathcal{M}^\infty$.

Proof outline.

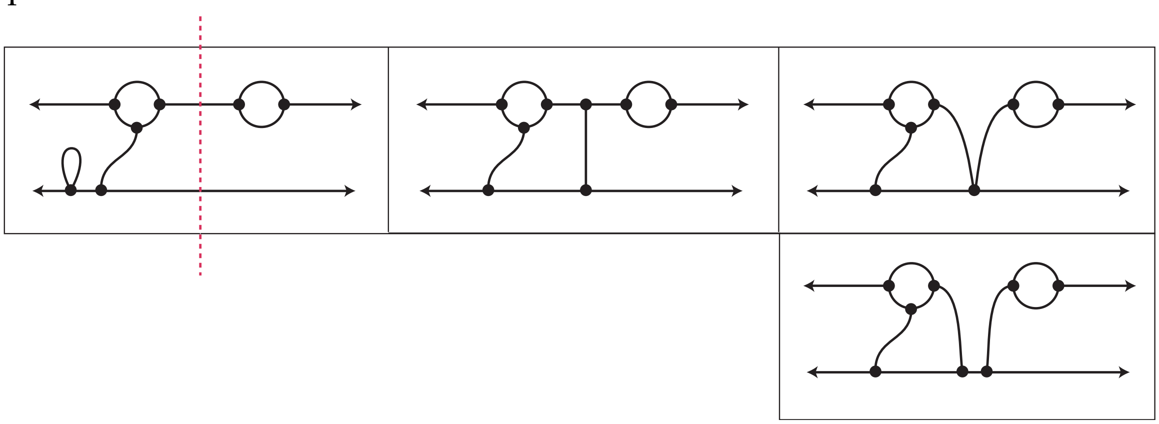
Add baseline:



Connect other components to baseline:



Split:



Do this splitting in many slices independently. Combine by partition of unity. Slide first and last pieces to $\pm\infty$ as before. Get $\mathcal{G}^{\infty,1} \simeq B\mathcal{M}^\infty$. □

The Group Completion Theorem gives a homology isomorphism

$$H_*(\mathbb{Z} \times \lim_n \mathcal{M}_n^\infty) \cong H_*(\Omega B \mathcal{M}^\infty)$$

Thus

$$\begin{aligned} H_*(\mathbb{Z} \times \lim_n B\mathrm{Aut}(F_n)) &\cong H_*(\Omega B \mathcal{M}^\infty) \\ &\cong H_*(\Omega \mathcal{G}^{\infty,1}) \qquad \text{by Fact 2.2} \\ &\cong H_*(\Omega^k \mathcal{G}^{\infty,k}) \qquad \text{for all } k > 1 \text{ by Fact 2.1} \\ &\cong H_*(\Omega^\infty \mathcal{G}^{\infty,\infty}) \end{aligned}$$

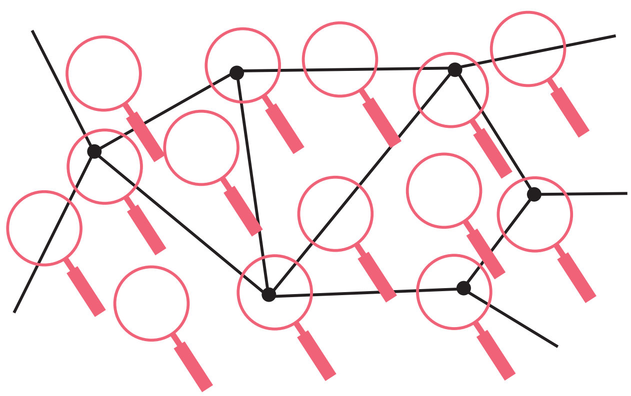
Taking one component, $H_*^{stab}\mathrm{Aut}(F_n) \cong H_*(\Omega_0^\infty \mathcal{G}^{\infty,\infty})$. End of Step 2.

Step 3

Fact 3: $\mathcal{G}^{N,N} \simeq S^N$

Hence $H_*^{stab}\mathrm{Aut}(F_n) \cong H_*(\Omega_0^\infty S^\infty)$

Idea: Rescale to a small ball about the origin to get a finite linear tree in a ball.



Technical point: special scanning lens needed for continuity:

Then shrink trees to their centerpoints.

The space of graphs with at most one point is S^N, the one-point compactification of $\mathbb{R}^N$.

This finishes Galatius' theorem.

The Barratt–Kahn–Priddy Theorem

This follows the same plan but is much easier. Take $\mathcal{G}^N$ to be the space of 0-dimensional graphs in $\mathbb{R}^N$, so $\mathcal{G}_n^\infty = B\Sigma_n$.

Again get $\mathcal{G}^{N,N} \simeq S^N$. This is why $H_*^{stab}\mathrm{Aut}(F_n) \cong H_*^{stab}\Sigma_n$.

The Madsen-Weiss Theorem

Take $\mathcal{G}^N =$ space of smooth compact surfaces in $\mathbb{R}^N$.

Step 1: $\mathrm{Diff}(S_g)$ has contractible components (g >1) so $\mathrm{MCG}(S_g) \simeq \mathrm{Diff}(S_g)$.

Thom: $B\mathrm{Diff}(S_g) =$ space of embedded $S_g \subset \mathbb{R}^\infty$.

Step 2: very similar to the $\mathrm{Aut}(F_n)$ case. One extra step needed for $\mathcal{G}^{N,1} \simeq \Omega\mathcal{G}^{N,2}$.

Step 3: $\mathcal{G}^{N,N} \simeq$ space of oriented affine 2-planes in $\mathbb{R}^N$. (Easy)

Handlebody Mapping Class Groups

Let $V_n =$ 3-dimensional handlebody of genus n.

- $\mathrm{MCG}(V_n) \longrightarrow \mathrm{Out}(F_n)$ is surjective. Large kernel.
- $\mathrm{MCG}(V_n) \longrightarrow \mathrm{MCG}(\partial V_n)$ is injective.
- $H_*\mathrm{MCG}(V_n)$ stabilizes (Hatcher–Wahl 2007)

Theorem $H_*^{stab} MCG(V_n) \cong H_*(\Omega_0^\infty S^\infty BSO(3)_+)$.

The "+" denotes adding a disjoint basepoint and gives $H_*^{stab}\mathrm{Out}(F_n)$ as a summand of $H_*^{stab} MCG(V_n)$.

Corollary $H_*^{stab}(MCG(V_n);\mathbb{Q}) \cong \mathbb{Q}[x_4, x_8, \cdots]$.

This is "half" of $H_*^{stab}(MCG(\partial V_n);\mathbb{Q}) \cong \mathbb{Q}[x_2, x_4, x_6, x_8, \cdots]$.

Remark: $\Omega^\infty S^\infty BSO(d)_+$ is the spectrum that arises when one scans general d-dimensional manifolds with boundary (Josh Genauer).

Idea of proof: Handlebodies are 3-dimensional thickenings of graphs, so enhance graphs with 3-dimensional tangential data.

Let $\mathcal{G}^N$ = space of *3-plane graphs*: graphs Γ in $\mathbb{R}^N$ as before, with a choice, at each point $x \in \Gamma$, of an oriented 3-plane P_x containing the tangent lines to the edges of Γ that contain x.

Step 1: $\mathcal{G}_n^\infty \simeq B\mathrm{MCG}(V_n)$ for $n > 1$. Uses 3-manifold facts:

- $\mathrm{Diff}(V_n)$ has contractible components for $n > 1$, so $\mathrm{Diff}(V_n) \simeq \mathrm{MCG}(V_n)$.
- The space of handle structures on V_n is contractible.

Step 2: Only minor modifications from the $\mathrm{Aut}(F_n)$ case.

Step 3: Also only minor modifications from the $\mathrm{Aut}(F_n)$ case. Get $\mathcal{G}^{N,N} \simeq$ one-point compactification of the space of pairs (P, x) where P is an oriented affine 3-plane in $\mathbb{R}^N$ and x is a point in P.

This is the Thom space of the trivial N-plane bundle over the Grassmann manifold $Gr^{N,3}$ of oriented 3-planes in $\mathbb{R}^N$, or equivalently, the N-fold suspension $S^N(Gr_+^{N,3})$. Let $N \to \infty$ to get $Gr_+^{\infty,3} = BSO(3)_+$.

Steps 2 and 3 work in higher dimensions as well. With a modified Step 1, one can prove:

<u>Theorem</u> $H_*^{stab} B\mathrm{Diff}(\#_n S^1 \times S^2) \cong H_*(\Omega_0^\infty S^\infty BSO(4)_+)$.

Idea: View $\#_n S^1 \times S^2$ as the boundary of a 4-dimensional handlebody.

- $\mathrm{Diff}(\#_n S^1 \times S^2)$ does not have contractible components, so no statement about MCG.
- Homology stability for $B\mathrm{Diff}(\#_n S^1 \times S^2)$ is unknown.