Strata and stabilizers of trees

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Goal of the talk

Outer space $\mathrm{CV}_N = \{$minimal free actions of $\mathbb{F}_N$ on simplicial trees$\}/\sim$.

Compactification $\overline{\mathrm{CV}}_N = \{$minimal *very small* actions on $\mathbb{R}$-trees$\}/\sim$.
Main example: action with trivial arc stabilizers.

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Applications

Give some kind of decomposition of any $T \in \overline{\mathrm{CV}}_N$ into simple building blocks.
Understand the stabilizer of T in $\mathrm{Out}(\mathbb{F}_N)$.

An example

α automorphism of $\langle a, b, c, d \rangle$:

$$\alpha : \begin{cases} \\ \\ a \mapsto ab \\ b \mapsto bab \end{cases}$$

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$$\#\{c, d\} \approx (1.6)^k$$
$$\wedge$$
$$\#\{a, b\} \approx (2.6)^k$$

successive images of d:

cad

dabcad

cadabbabdabcad

dabcadabbabbababbabcadabbabdabcad

cadabbabdabcadabbabbababbabbababbababbabbababbabdabcadabbabbababbabcadabbabdabcad

dabcadabbabbababbabcadabbabdabcadabbabbababbabbababbababbabbababbabbababbababbabbababbababbabbababbabbababbababbabbababbabcadabbabdabcadabbabbababbabbababbababbabbababbabdabcadabbabbababbabcadabbabdabcad

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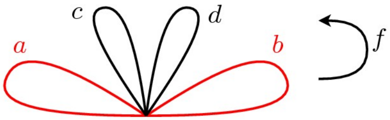

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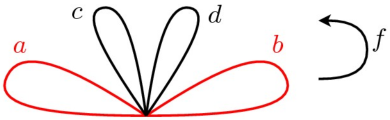
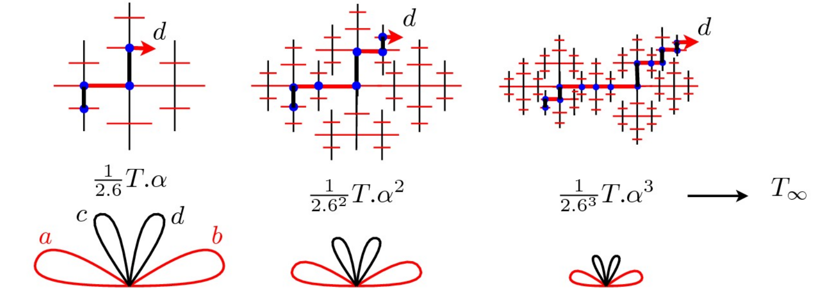


successive images of the path d, rescaled by 2.6^k

Tree interpretation: axis of the element d on $\frac{1}{2.6^k} T.\alpha^k$.



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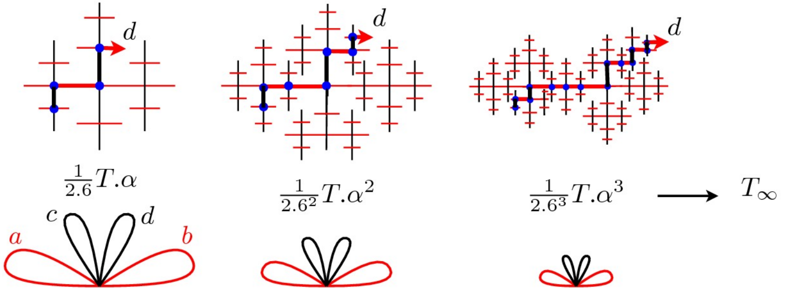

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Facts

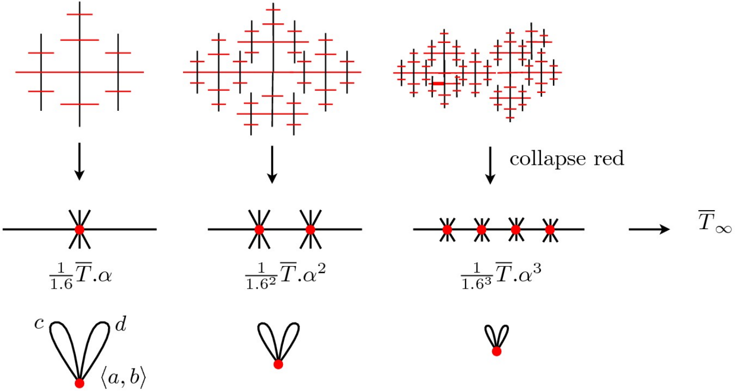
- T_∞ is α-invariant: there exists an α-equivariant homothety $H_\alpha : T_\infty \to T_\infty$
- $\langle a, b \rangle$ preserves a subtree $Y \subset T_\infty$, Y is H_α-invariant.
- Y is closed and disjoint from its translates

One can collapse Y equivariantly and get a topological $\mathbb{R}$-tree, with an action of F_N:

$$Y \hookrightarrow T_\infty \twoheadrightarrow T/Y$$

Other description of the collapsed tree: $T/Y = \overline{T}_\infty$.

Collapse all red edges before taking limit:



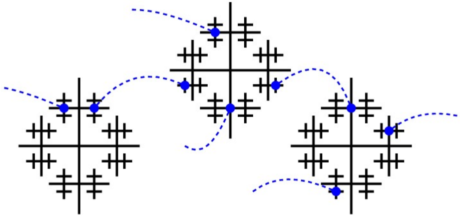
Theorem [G-Levitt]

Any $T \in \overline{\mathrm{CV}}_N$ can be obtained from simplicial trees and *mixing* trees by iterating two constructions:

- extensions $Y \hookrightarrow T \twoheadrightarrow T/Y$
- graph of actions

Mixing: minimality condition $\Rightarrow$ every orbits meets every segment in a dense set.

Graph of actions = Free amalgamated product of actions on $\mathbb{R}$-trees, glued along points.



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Remark: this obliges to consider topological $\mathbb{R}$-trees, with (non-nesting) actions by homeomorphisms.
If mixing, such topological actions have an invariant metric.

Admissible subtrees

To simplify, assume T has no simplicial arc (branch points are dense), arc stabilizers are trivial.

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Example 3. T is mixing if and only if it has no admissible subtree.

Main finiteness result

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There are only finitely many orbits of admissible subtrees $Y \subset T$.

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Next goal

Use this theorem to understand the $\mathrm{Out}(F_N)$-stabilizer of T.

Projective stabilizer $\mathrm{Aut}([T]) =$
set of $\alpha \in \mathrm{Aut}(F_N)$ s.t. $\exists$ α-equivariant *homothety* $H_\alpha : T \to T$.

Isometric stabilizer: $\mathrm{Aut}(T) =$
set of $\alpha \in \mathrm{Aut}(F_N)$ s.t. $\exists$ α-equivariant *isometry* $H_\alpha : T \to T$.

$\mathrm{Out}([T])$ and $\mathrm{Out}(T)$ = their images in $\mathrm{Out}(F_N)$.

Stabilizer of a simplicial tree

Γ a graph of groups, $T = \tilde{\Gamma}$ Bass-Serre tree.

General facts:

1. $\mathrm{Out}_0(\tilde{\Gamma}) \subset \mathrm{Out}(\tilde{\Gamma})$ finite index subgroup acting trivially on Γ.
2. There is a map $\rho : \mathrm{Out}_0(\tilde{\Gamma}) \to \prod_v \mathrm{Out}(G_v)$
3. Dehn twists are in the kernel of ρ
4. Elements of $\mathrm{Out}(G_v)$ which act like a conjugation on each edge group are in the image of ρ

Def: McCool group

Fix $\{E_1, \dots E_n\}$ some subgroups in free group F_k. The set of automorphisms $\alpha \in \mathrm{Out}(F_k)$ acting like a conjugation on each E_i is a *McCool* group.

Theorem (G-Levitt)

Fix $T \in \overline{\mathrm{CV}}_N$.

- $\mathrm{Out}(T)$ *has a finite index subgroup* $\mathrm{Out}_0(T)$ *s.t.*

$$1 \to \prod \textit{free groups} \to \mathrm{Out}_0(T) \to \prod \textit{McCool gps} \to 1$$

- *The set of scaling factors of* $\mathrm{Out}([T])$ *is a cyclic subgroup of* $\mathbb{R}_+^*$ *[Lustig]*

Remark: the McCool groups are McCool groups of point stabilizers. The free groups correspond to Dehn twists.

Proposition

McCool groups virtually have a finite classifying space.

Corollary

So does the stabilizer of T in $\mathrm{Out}(F_N)$.

Proof

Idea: construct a simplicial tree $\tilde{\Gamma}$ on which $\mathrm{Out}(T)$ acts.

1. All automorphisms α in some finite index subgroup of $\mathrm{Out}_0(T) \subset \mathrm{Out}(T)$ are *piecewise-F_N*.
2. $\mathrm{Out}_0(T)$ is *uniformly* piecewise-F_N: there exists a piecewise decomposition of T that is compatible with every $\alpha \in \mathrm{Out}_0(T)$.
3. There is a simplicial tree $\tilde{\Gamma}$ dual to this piecewise decomposition
4. $\mathrm{Out}_0(T)$ occurs as an extension of McCool groups by Dehn twists in Γ.