Definable and negligible subsets of free groups

(in honor of Karen Vogtmann's 60th birthday)

joint with Mladen Bestvina

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OUTLINE

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1. EXTENSION PROBLEMS

- $\mathbb{F}$ is non-abelian free group with finite basis B.

- All groups are fg.

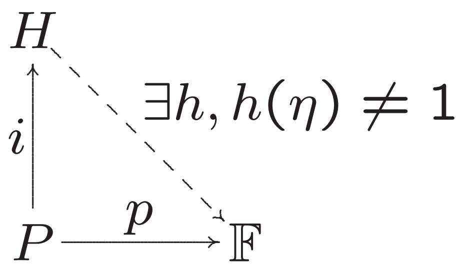
Problem 1 (0-quantifier). Describe $Hom(P, \mathbb{F})$.

Example. If P is a rank n free group, then $Hom(P, \mathbb{F}) \cong \mathbb{F}^n$.

Problem 1 was solved by Myasnikov-Kharlampovich, Sela. Later we describe a solution.

Problem 2 (E-Problem). Given $i : P \to H$ and finite subset $\overrightarrow{\boldsymbol{\eta}}$ of H, describe the set of homomorphisms $p : P \to \mathbb{F}$ that admit an extension h to H killing no $\eta \in \overrightarrow{\boldsymbol{\eta}}$.

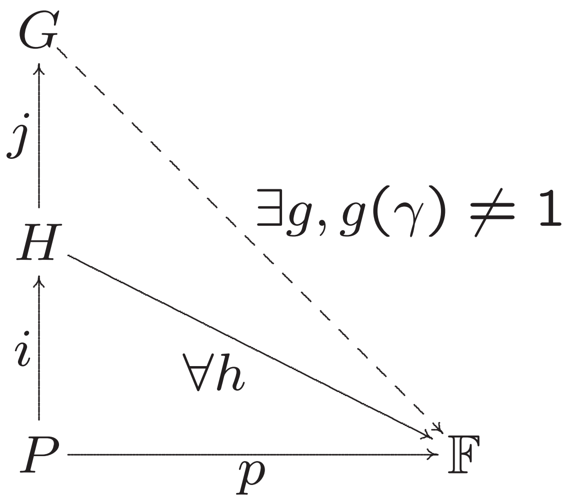
$$E := \{p \in Hom(P, \mathbb{F}) \mid \exists h \text{ such that } h(\eta) \neq 1 \text{ for } \eta \in \overrightarrow{\boldsymbol{\eta}}\}$$



Example. If $P = H$ and $\overrightarrow{\boldsymbol{\eta}} = \emptyset$ then $E = Hom(P, \mathbb{F})$.

Example. If $1 \in \overrightarrow{\boldsymbol{\eta}}$ then $E = \emptyset$.

Problem 3 (AE-Problem). Given $P \xrightarrow{i} H \xrightarrow{j} G$ and finite subset $\overrightarrow{\gamma}$ of G, describe the set AE of p with that property that all extensions h admit a further extension g killing no $\gamma \in \overrightarrow{\gamma}$.



• The most general AE-problem allows finitely many $j : H \to G_j$, but we will ignore this for this talk.

EAE-**Problem,** $AEAE$-**Problem, . . .**

2. DEFINABLE SUBSETS OF $\mathbb{F}$

• A *definable set* is a subset of $Hom(P, \mathbb{F})$ that is in the Boolean algebra generated by $EAE...$-sets.

• We restrict to the case $P = \mathbb{Z}$ and so $Hom(P, \mathbb{F})$ is identified with $\mathbb{F}$. Hence, we consider definable subsets of $\mathbb{F}$.

• We have seen $\emptyset$ and $\mathbb{F}$ are definable. Our goal is to show that up to some equivalence these are the only examples.

Theorem (Sela). The Boolean algebra generated by AE-sets contains all definable sets.

3. NEGLIGIBLE SETS AND THE CONJECTURE

• $\mathbb{F}$ is a free group with finite basis B.

• A *piece* of a word $w \in \mathbb{F}$ is a non-trivial subword that appears in two different ways.

Example. • ab is a piece of $abcBA$

• a^2 is a piece of a^3

• a^3 is not a piece of a^3

• A subset N of $\mathbb{F}$ is *negligible* if there is $\epsilon > 0$ such that all but finitely many $w \in N$ have a piece with relative length $\geq \epsilon$, i.e.

$$\frac{length(piece)}{length(w)} \geq \epsilon$$

• Complements of negligible sets are *co-negligible*.

Example. • Finite sets are negligible and subsets of negligible sets are negligible.

• $\{ba^i \mid i \in \mathbb{N}\}$ is negligible. (Any $0 < \epsilon < 1$ works.)

Conjecture. Definable subsets of $\mathbb{F}$ are negligible or co-negligible.

• A sequence $\{w_i\}$ in $\mathbb{F}$ is a *test sequence* if the relative lengths of pieces of w_i goes to 0.

Example. $\mathbb{F}$ contains the test sequence $\{baba^2ba^3 \cdots ba^ib\}_i$. More generally, a set containing a coset of a non-abelian subgroup of $\mathbb{F}$ contains a test sequence.

• A subset of $\mathbb{F}$ is negligible iff it doesn't contain a test sequence.

• The set $Prim$ of primitive elements of $\mathbb{F}$ is neither negligible nor co-negligible if $rank(\mathbb{F}) > 2$. (If $B = \{a, b, c\}$ then $Prim$ contains $c\langle a, b\rangle$ and the complement of $Prim$ contains $\langle [a, b], c[a, b]C\rangle$.) ($\stackrel{Conj}{\Longrightarrow}$ Rips-Sela question.)

• A proper non-abelian subgroup of $\mathbb{F}$ is neither negligible nor co-negligible. ($\stackrel{Conj}{\Longrightarrow}$ Mal'cev question.)

4. SHORTEST FAILURES

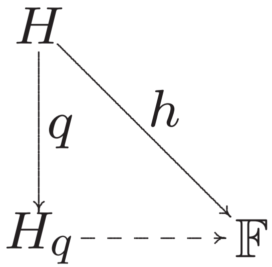
• Our approach to the conjecture is via *shortest failures*. We give an example of this technique in a solution to Problem 1.

Problem 1. Describe $Hom(H, \mathbb{F})$.

• H is a *limit group* if for all finite subsets $\overrightarrow{\boldsymbol{\eta}}$ of H there is $h : H \to \mathbb{F}$ that is injective on $\overrightarrow{\boldsymbol{\eta}}$.

• A *factor set of H* is a finite set $\overrightarrow{\boldsymbol{q}} = \{q : H \to H_q\}$ of proper epimorphisms.

• $h : H \to \mathbb{F}$ *factors through* $\overrightarrow{\boldsymbol{q}}$ if there is $q \in \overrightarrow{\boldsymbol{q}}$ such that h factors through q.



• H is not a limit group iff there is a factor set $\overrightarrow{\boldsymbol{q}} = \{q : H \to H_q\}$ for H such that all $h : H \to \mathbb{F}$ factor through $\overrightarrow{\boldsymbol{q}}$.

• With a little extra work, we may take each H_q to be a limit group.

• h *factors through* $\overrightarrow{\boldsymbol{q}}$ *up to twisting* if there is $a \in Aut(H)$ such that $h \circ a$ factors through $\overrightarrow{\boldsymbol{q}}$.

Theorem (Kharlampovich-Myasnikov, Sela). Suppose H is not free. Then there is a factor set $\overrightarrow{\boldsymbol{q}} = \{q : H \to H_q\}$ such that all $h : H \to \mathbb{F}$ factor through $\overrightarrow{\boldsymbol{q}}$ up to twisting. Further, we may take each H_q to be a limit group.

Idea of Proof*. • We may assume H is a limit group.

• Let $\eta_1, \eta_2, \ldots$ be an enumeration of the non-trivial elements of H. Let $\overrightarrow{\boldsymbol{q}}_i$ be the factor set $\{H/\eta_1, \ldots, H/\eta_i\}$.

• Let h_i be a shortest failure to the statement for $\overrightarrow{\boldsymbol{q}}_i$ where the length of h_i is the sum of the lengths in $\mathbb{F}$ of the images of a generators for H.

*details can be found in our paper *Notes on Sela's work*.

• $h : H \to \mathbb{F}$ gives an action T_h of H on the Cayley graph (tree) $T_{\mathbb{F}}$ of $\mathbb{F}$.

• The projectivized space of real H-trees is compact (Culler-Morgan).

• $T_H = \lim T_{h_i}$ is faithful.

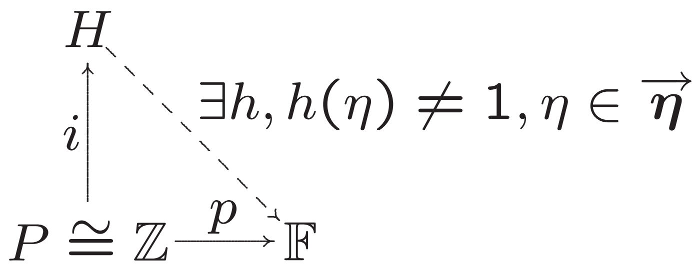
• There is a structure theory for faithful T_H-trees of this type. In particular, there is $a \in Aut(H)$ such that $|h_i \circ a| < |h_i|$ for $i \gg 0$ (Shortening).

• So $h_i \circ a$ and hence h_i factors through $\overrightarrow{\boldsymbol{q}}_i$ up to twisting, contradiction. $\square$

Theorem (Sela). (Limit group induction) A sequence of epimorphisms between limit groups stabilizes.

5. THE CONJECTURE FOR E-SETS

• To see where negligibility arises, we give the idea of the proof of the conjecture in a special case. Recall the definition of E.

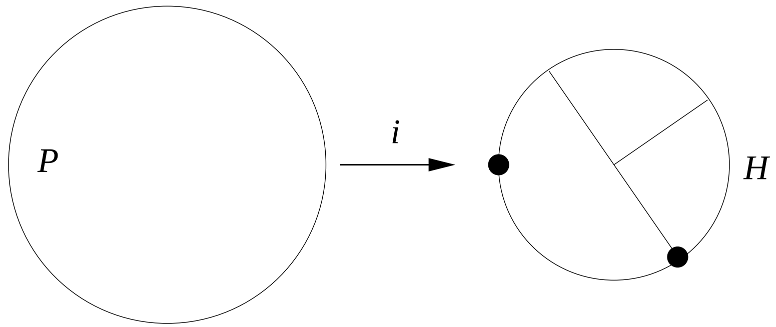


Proposition. Either E is negligible or it contains $\mathbb{F} \setminus \{1\}$.

Idea of Proof. Suppose $\{p_i\}$ is a test sequence in E. Let h_i be a shortest extension of p_i. $T_H = \lim T_{h_i}$.

• If T_H is not faithful, then h_i factors through a quotient H_q for large i and we are done by limit group induction. ($E(P \to H_q)$ contains the same test sequence.)

• For convenience, suppose T_H is simplicial with trivial edge stabilizers. Consider $T_P/P \to T_H/H$.



• If the image misses an edge then $H = A * B$ with $i(P) \subset A$. It follows that h_i is not shortest for large i. (In particular, P is not elliptic in T_H.)

- If the image crosses an edge more than once then p_i is not a test sequence.

- Conclude $H = P * H_0$ and this case is an exercise. $\square$