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On Lie groups whose Dehn function is polynomial

(joint with R. Tessera)

G connected Lie group (with left-inv. Riem. metric) → Dehn function $\delta(n)$
→ Asymptotic cone

<u>Def</u> $\delta(n) = \sup \{ \text{area}(\gamma) \mid \gamma \text{ null-homotopic loop of length} \leq n \}$

$\text{area}(\gamma) = \inf \{ \text{area}(D) \mid D \text{ disc}, \partial D = \gamma \}$.

Informally, $\text{Cone}(G) = $ "$\lim_{n \to \infty} \frac{1}{n} G$" $\quad \left(\frac{1}{n} G = (G, \frac{1}{n} d) \right.$ d Riem. distance$)$

Formally $\text{Cone}_\omega(G) = \left(\{ (x_n) \mid x_n \in G, \left(\frac{1}{n} d(x_0, x_n)\right) \text{bounded} \} / \sim , d_\omega \right)$

$(x_n) \sim (y_n)$ if $d_\omega((x_n), (y_n)) = 0$

$d_\omega((x_n), (y_n)) = \lim_\omega \frac{1}{n} d(x_n, y_n)$ $\qquad \omega$ non-principal ultrafilter

- — the asymptotic behaviour of $\delta(n)$
  — the bilipschitz [hence topological] type of $\text{Cone}_\omega(G)$

  are quasi-isometry invariants of G. (In particular, do not depend on choice of left-inv. Riem. metric)

Question — when is $\delta(n)$ polynomial / exponential
— when is $\text{Cone}_\omega(G)$ simply connected?

In the sequel, I'll define a <u>partition</u> of the class of connected Lie groups into four classes, so that

<u>Theorem</u> (CT) Let G be a connected Lie group. Depending on the class in which it lies, we have

class	Dehn function $\delta(n)$	$\mathrm{Cone}_\omega G$	
(P)	polynomial ($\preccurlyeq n^s$ for some s)	$\pi_1 = \{1\}$	π_1 abelian
($C_{\mathbb{Q}}$)		π_1 uncountable	
($C_{\mathbb{R}}$)	exponential		
(S)			$\pi_1 \supseteq F_2$ non abelian free group

<u>Rk</u>

- All four classes contain groups with polycyclic lattices, for which the results also hold (by QI-invariance)
- Alternative $\delta(n)$ polynomial / exponential
- Alternative $\pi_1 \mathrm{Cone}_\omega G$ abelian / with free subgroups (independently on ω)
- All previous known examples lie in $(P) \sqcup (S)$

– Examples in the class (P)

* G nilpotent of class s $\Rightarrow$ $\delta(n) \preccurlyeq n^{s+1}$
  (sharp for free s-nilpotent groups)

  (precise behaviour = hard problem even for some examples with s=2)

  $\Rightarrow$ $\mathrm{Cone}_\omega G \underset{\text{homeo.}}{\approx} \mathbb{R}^d$ (Pansu)

* G hyperbolic à la Gromov $\Leftrightarrow$ $\delta(n) \approx n$ ($\Leftrightarrow$ $\delta(n) \not\succcurlyeq n^2$)
  $\Rightarrow$ $\mathrm{Cone}_\omega G$ $\mathbb{R}$-tree

  examples: S simple of rank 1, $\mathbb{R} \ltimes N$, contracting action of $\mathbb{R} \curvearrowright N$.

* G CAT(0) (up to QI) $\Rightarrow$ $\delta(n) \approx n^2$ (unless hyperbolic)
  $\Rightarrow$ $\mathrm{Cone}_\omega G$ CAT(0) (up to bilip.)

* Many other solvable groups, e.g. Upper SOL-groups:
  $USOL^d := \mathbb{R}^{d-1} \ltimes \mathbb{R}^d$ acting by diagonal matrices of det = 1, if $d \geqslant 3$ (Gromov)

for this example:
  $\longrightarrow$ $\delta(n) \approx n^2$ (but some higher-dimensional Dehn function is exponential — filling of (d-1)-spheres)

  $\longrightarrow$ $\pi_1 \mathrm{Cone}_\omega G = \{1\}$ (but $\pi_{d-1} \neq \{1\}$).

Rk. it is hard in general to determine the precise growth rate of $\delta(n)$, even for G nilpotent.

— Examples in (S) $\lambda > 0$

- $SOL_\lambda = \mathbb{R}^2 \rtimes \mathbb{R}$ $\quad t\cdot(x,y) = \begin{pmatrix} e^t & \\ & e^{-\lambda t}\end{pmatrix}\begin{pmatrix} x \\ y\end{pmatrix}$

($SOL_1 = SOL$ is unimodular and has lattices and thus occurs in classification of compact 3-dim. manifolds)

⟶ $\delta(n) \approx e^n$ (Epstein, Thurston for $\lambda=1$, Gromov)

⟶ $\pi_1 \mathrm{Cone} \supseteq F_2$ (Gromov, Burillo)

- More generally, groups surjecting onto some SOL_λ
- $SL_2\mathbb{R} \ltimes \mathbb{R}^2$ (has cocompact subgroup mapping onto SOL_2)
- $N \rtimes \mathbb{R}$ whenever the action of $\mathbb{R}$ on the nilpotent group N distorts N exponentially, unless the action is contracting (hyperbolic case).

Remark:

any metabelian connected Lie group lies in (P) ∪ (S) (more generally, any solvable connected Lie group whose <u>exponential radical</u> is abelian, see p.(8))

— Examples in $(C_{\mathbb{R}})$ ("C" for Central)

- $G = Sp_4(\mathbb{R}) \ltimes \mathbb{R}^4$ (standard action)

idea: the invariant symplectic form on $\mathbb{R}^4$ gives rise to an extension

$$1 \longrightarrow \mathbb{R} \xrightarrow{i} \widetilde{G} \longrightarrow G \longrightarrow 1$$

where * $\widetilde{G} = Sp_4(\mathbb{R}) \ltimes Heis_5$

$Heis_5$ 5-dimensional Heisenberg group, whose center (= derived subgroup $= i(\mathbb{R})$) is one dimensional

* $i(\mathbb{R})$ is central and exponentially distorted in $\widetilde{G}$: the length of $i(e^n)$ in $\widetilde{G}$ is $\approx n$. This causes $\delta(n) \succcurlyeq e^n$ for G ($\delta(n) \preccurlyeq e^n$ for an arbitrary connected Lie group is an observation of Gromov).

* This extension gives rise to maps

$$\mathrm{Cone}_\omega(\mathbb{R}, \text{log-metric}) \longrightarrow \mathrm{Cone}_\omega(\widetilde{G}) \longrightarrow \mathrm{Cone}_\omega(G)$$

which behaves like a covering (existence and uniqueness of lifting of paths), yielding that $\pi_1 \mathrm{Cone}\, G$ is uncountable.

* By the theorem, $\widetilde{G}$ is in (P) so this computes:

$$\pi_1 \mathrm{Cone}_\omega G \simeq \underbrace{\mathrm{Cone}_\omega(\mathbb{R}, \text{log-metric})}_{\text{abelian group}}.$$

- Let $\tilde{G}$ be Abels' group : real matrices of the form

$$\begin{pmatrix} 1 & * & * & \circledast \\ & * & * & * \\ & & * & * \\ & & & 1 \end{pmatrix} \leftarrow \text{central.}$$

Then $\tilde{G}$ lies in (P), its center Z is 1-dim, and is exponentially distorted hence $G = \tilde{G}/Z$ lies in $(C_{\mathbb{R}})$.

— Example in $(C_{\mathbb{Q}})$

- $G = SL_3\mathbb{R} \ltimes \mathbb{R}^8$ ↳ $= \underline{sl_3}\mathbb{R}$ with adjoint action.

Def. A <u>triangulable group</u> is a closed connected subgroup of real upper triangular matrices

<u>Fact</u> Every connected Lie group G is QI ("topologically commensurable" in a suitable sense) to a triangular group G_1. Often (e.g. if G is algebraic), G_1 can be taken as a closed cocompact subgroup.

Example : $G = SL_d\mathbb{R} \ltimes \mathbb{R}^d \supseteq \underbrace{T_d}_{\text{upper triang. mat. in } SL_d}\mathbb{R} \ltimes \mathbb{R}^d$

⟶ Will define the classes $\mathcal{C} = (S), (C_\mathbb{R}), (C_\mathbb{Q}), (P)$ and then say that G is in $\mathcal{C}$ if G_1 is. ← <u>for triangulable groups.</u>

<u>Weights</u>

D group, V f.dim real vector space on which D acts diagonalizably with positive eigenvalues. There is a common diagonalization

$$V = \bigoplus_{\alpha \in \mathrm{Hom}(D,\mathbb{R})} V_\alpha$$

$$V_\alpha = \{ v \in V \mid \forall d \in D \quad d \cdot v = e^{\alpha(d)} v \}$$

common eigenspace, called "weight space".

α for which $V_\alpha \neq \{0\}$ are called <u>weights</u>.

- Let G be a triangulable group.

Consider the adjoint action of G on its Lie algebra $\mathfrak{g}$. This action is triangulable. Choose a triangulation of this action, and get a new, diagonalizable action, by replacing nonzero coefficients by 0: $G \underset{diag}{\curvearrowright} \mathfrak{g}$, thus getting a decomposition $\mathfrak{g} = \bigoplus_{\alpha \in Hom(G,\mathbb{R})} \mathfrak{g}_\alpha$.

The new action is also by Lie algebra ~~…~~ automorphisms whence $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subseteq \mathfrak{g}_{\alpha+\beta}$ i.e.

$(\mathfrak{g}_\alpha)$ is a grading of $\mathfrak{g}$ in the vector space $Hom(G,\mathbb{R})$.

Def The Lie subalgebra $\mathfrak{n}$ generated by $\bigoplus_{\alpha \neq 0} \mathfrak{g}_\alpha$ is an ideal, called exponential radical of $\mathfrak{g}$, and does not depend on choices ($\mathfrak{g}/\mathfrak{n}$ is the largest nilpotent quotient of $\mathfrak{g}$).

Def
- the weights of $G \underset{diag}{\curvearrowright} \mathfrak{n}$ are called weights of G
- the weights of $G \underset{diag}{\curvearrowright} \mathfrak{n}/[\mathfrak{n},\mathfrak{n}]$ are called principal weights of G.

Rk. 0 is not a principal weight

Def (Abels) G is in (S) if $\exists\, \alpha, \beta$ principal weights s.t. 0 lies in the segment $[\alpha\ \beta] \subseteq Hom(G,\mathbb{R})$.

Otherwise G is tame.

Digression $\underline{n}$ Lie algebra.

$$\underline{n} \wedge \underline{n} \wedge \underline{n} \xrightarrow{d_3} \underline{n} \wedge n \xrightarrow{d_2} \underline{n}$$

$$x \wedge y \mapsto [x, y]$$

$$x \wedge y \wedge z \longmapsto x \wedge [y, z] + y \wedge [z, x] + z \wedge [x, y] \qquad d_2 \circ d_3 = 0$$

$$H_2(n) := \mathrm{Ker}(d_2) / \mathrm{Im}(d_3)$$ second homology group

If $\underline{n}$ graded in an abelian group, d_i preserves the graduation, so $H_2(\underline{n})$ is graded.

Turn back to the notation above.

$\underline{\text{Def}}$ $\mathfrak{g}$ is in $(C_{\mathbb{R}})$ is $\mathfrak{g}$ is tame and $H_2(n)_0 \neq 0$

$\underline{\text{Def}}$ $\mathfrak{g}$ is in $(C_{\mathbb{Q}})$ if $\mathfrak{g}$ is tame, $H_2(n)_0 = 0$ but $H_2(n|\mathbb{Q})_0 \neq 0$

($n|\mathbb{Q}$ is n viewed as an uncountable-dimensional Lie algebra over $\mathbb{Q}$ by restriction of scalars).

$\underline{\text{Def}}$ $\mathfrak{g}$ is in (P) if $\mathfrak{g}$ is tame and $H_2(n|\mathbb{Q})_0 = 0$.