The cohomology of $\mathrm{Out}(F_r)$ and the Eichler-Shimura isomorphism

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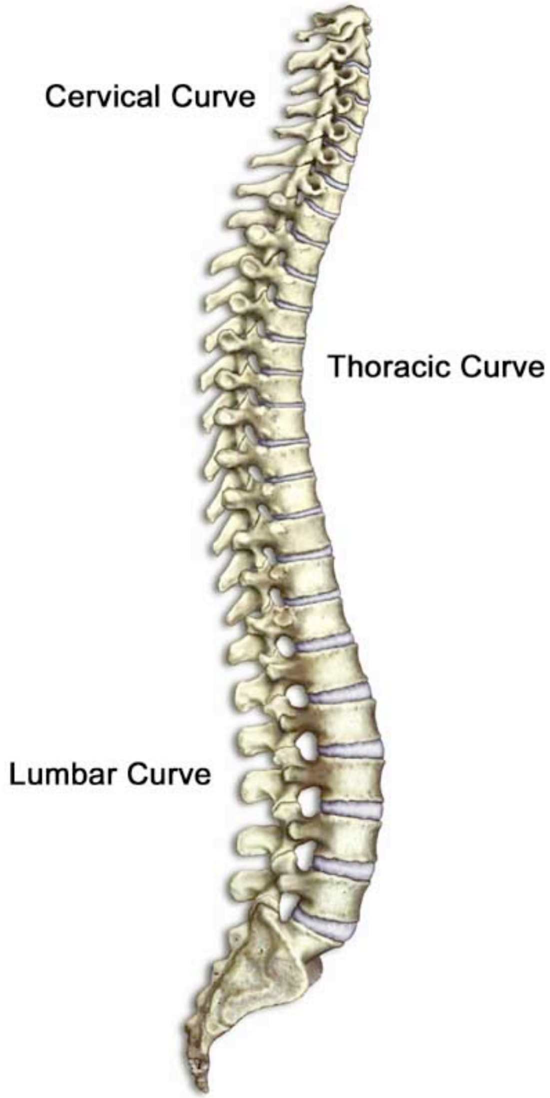
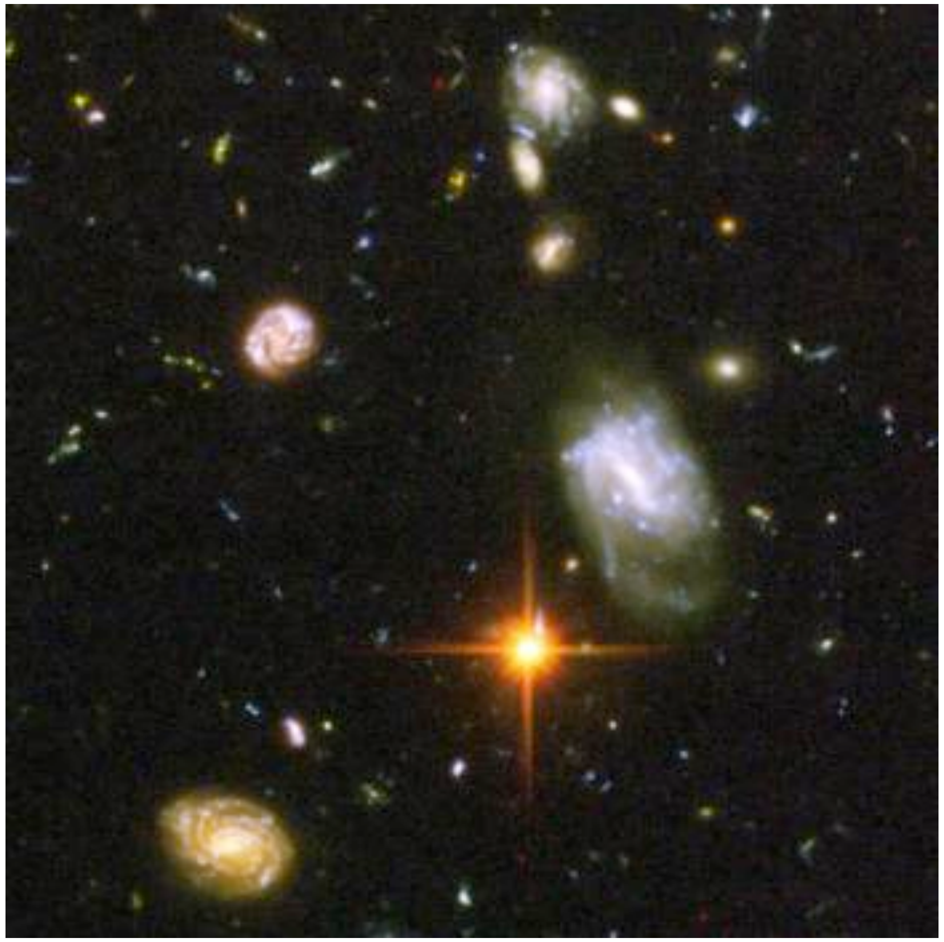
June 24, CIRM

joint with Martin Kassabov and Karen Vogtmann


Cervical Curve
Thoracic Curve
Lumbar Curve



Happy Birthday, Karen!

Moving to the Lie category

$$PH^*(\ell_\infty^+)^{\mathfrak{sp}}$$

$$\bigoplus_{r=2}^{\infty} H_*(Out(F_r);\mathbb{Q})$$

sp-invariant theory
(Kontsevich,
Conant-Vogtmann)

$\cong$

$\cong$

spectral sequence
on spine of outer
space

Lie graph homology

$$PH^*(\ell_\infty^+)^{\mathfrak{sp}} \cong \bigoplus_{r=2}^{\infty} H_*(Out(F_r);\mathbb{Q})$$

Moving to the Lie category

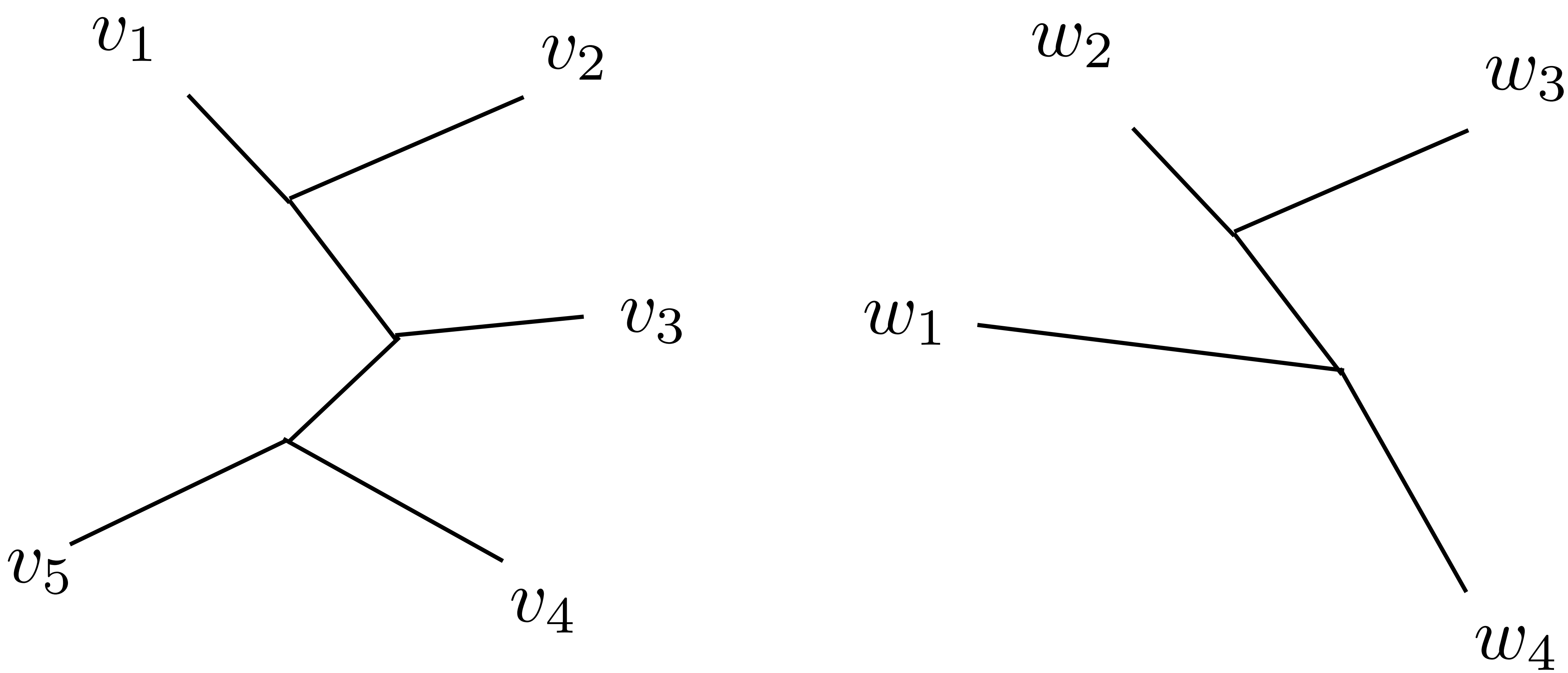
$$\begin{tikzcd}[column sep=large] PH^*(\ell_\infty^+)^{\mathfrak{sp}} \arrow[dr, leftrightarrow, "\cong", "\substack{\text{sp-invariant theory}\\ \text{(Kontsevich,}\\ \text{Conant-Vogtmann)}}"'] & & \bigoplus_{r=2}^{\infty} H_*(Out(F_r);\mathbb{Q}) \\ & \text{Lie graph homology} \arrow[ur, leftrightarrow, "\cong", "\substack{\text{spectral sequence}\\ \text{on spine of outer}\\ \text{space}}"'] & \end{tikzcd}$$

$$PH^*(\ell_\infty^+)^{\mathfrak{sp}} \cong \bigoplus_{r=2}^{\infty} H_*(Out(F_r);\mathbb{Q})$$

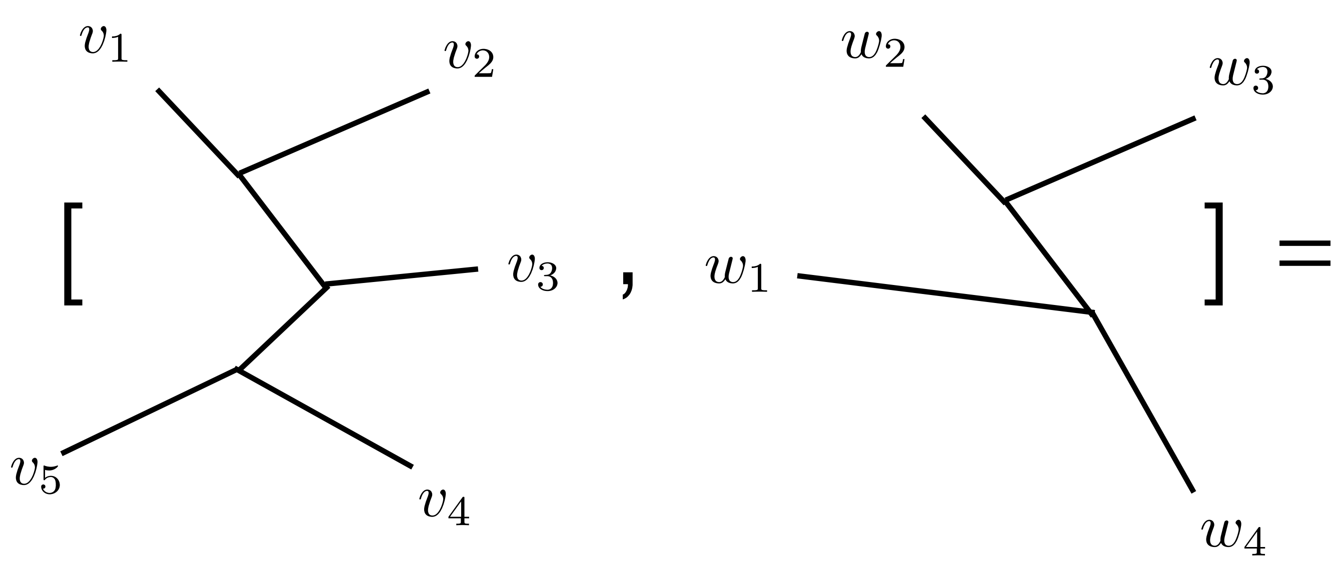
So we study the homology of Out via the Lie algebra ℓ_∞^+.

Lie Spiders

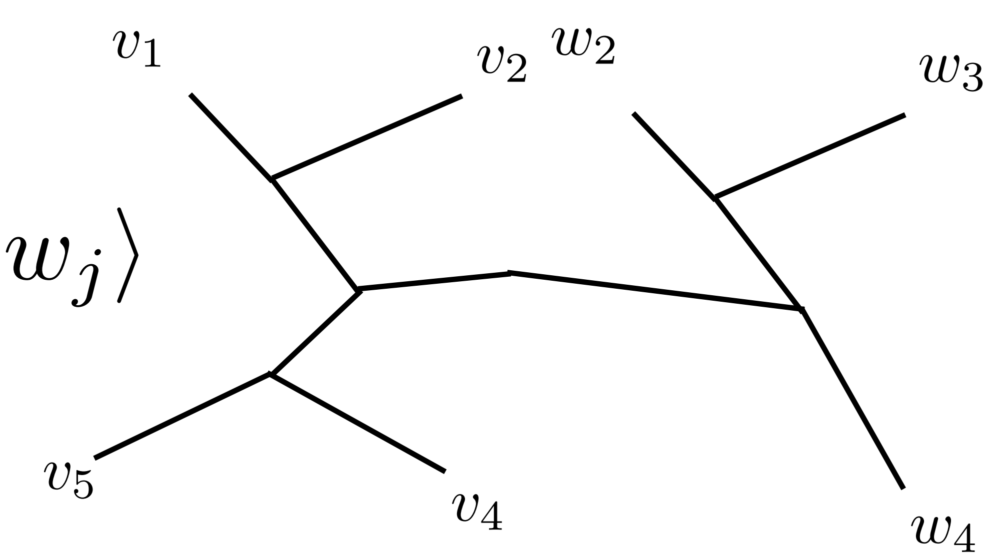
Let $(V, \langle\, ,\ \rangle)$ be a vector space with a nondegenerate bilinear form.



Modulo Jacobi (IHX) and antisymmetry.



$$\sum_{i,j}\langle v_i, w_j\rangle$$



Antisymmetry: [x,y]=-[y,x] $\Rightarrow \langle\ ,\ \rangle$ is symplectic.

Jacobi Identity: $\Leftarrow$ Generalized associativity (cyclic operad structure)

Let V_n be a fixed standard 2n-dimensional symplectic vector space.

ℓ_n^+ is the Lie algebra of spiders labeled by V_n, with at least 3 legs.

$$\ell_\infty^+ = \lim_{n \to \infty} \ell_n^+$$

Utility of the abelianization

$$\mathfrak{g} \longrightarrow \mathfrak{a}$$

$$H^*(\mathfrak{a}) \longrightarrow H^*(\mathfrak{g})$$

$$\Lambda^*(\mathfrak{a}) \longrightarrow H^*(\mathfrak{g})$$

$$\Lambda^*(\mathfrak{a})^{\mathfrak{sp}} \longrightarrow H^*(\mathfrak{g})^{\mathfrak{sp}}$$

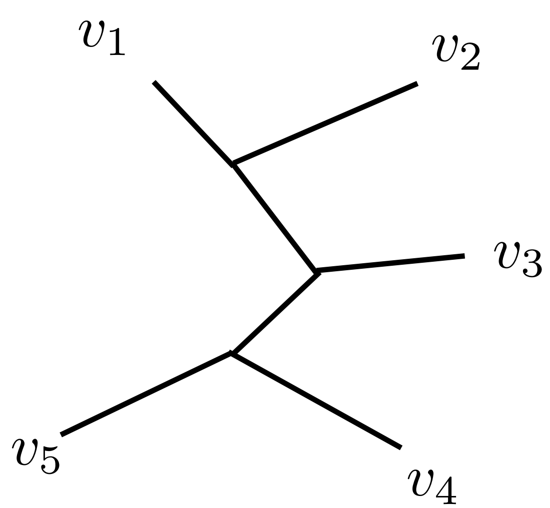
In some cases, the kernel is not too large.

Morita constructed a surjective Lie algebra map

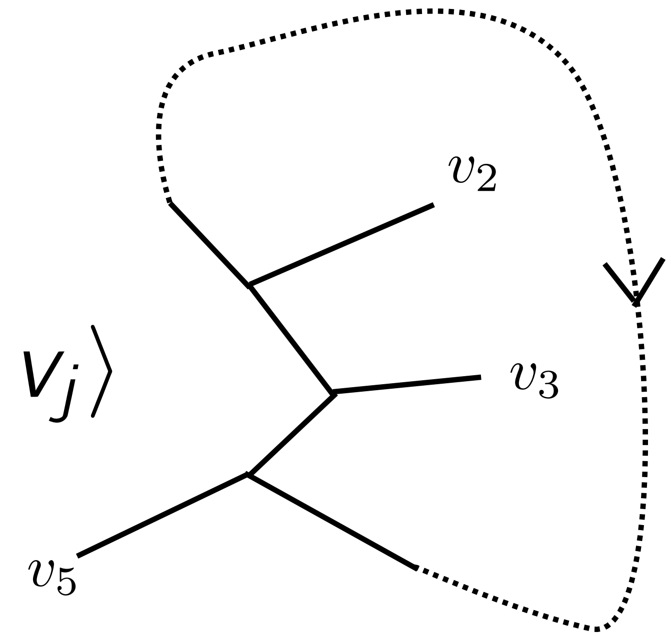
$$\ell_\infty^+ \twoheadrightarrow \Lambda^3 V \oplus \bigoplus_{k=1}^{\infty} S^{2k+1} V$$

abelian Lie algebra

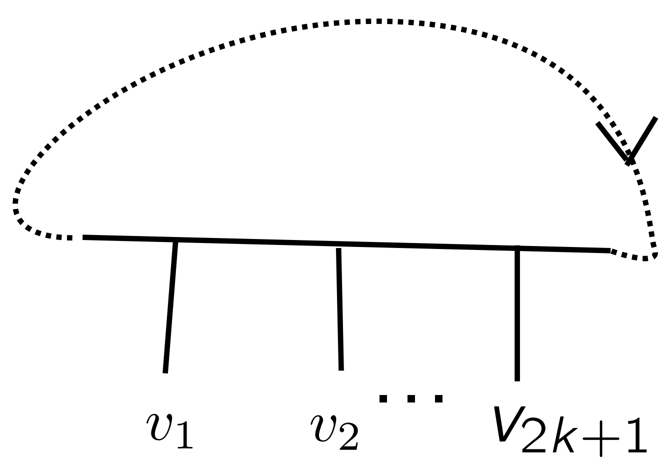
He conjectured that this is precisely the abelianization.



$$\xrightarrow{tr} \sum_M \langle v_i, v_j \rangle$$



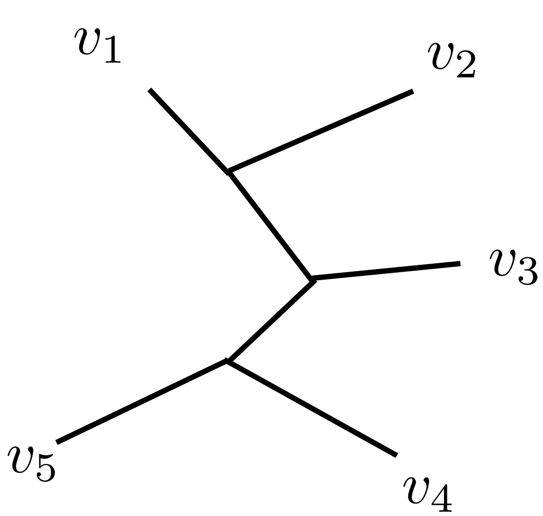
Not hard: tr vanishes on nontrivial brackets.



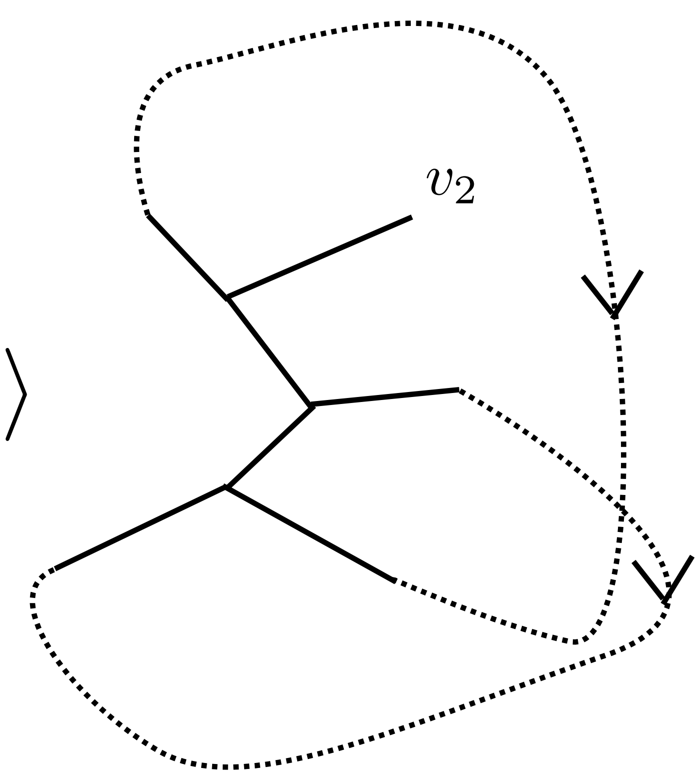
$$\longleftrightarrow \quad S^{2k+1}V$$

Idea: generalize Morita’s trace.

$$Tr = \exp(tr) \colon \Lambda\ell_\infty^+ \to \mathcal{HG}$$

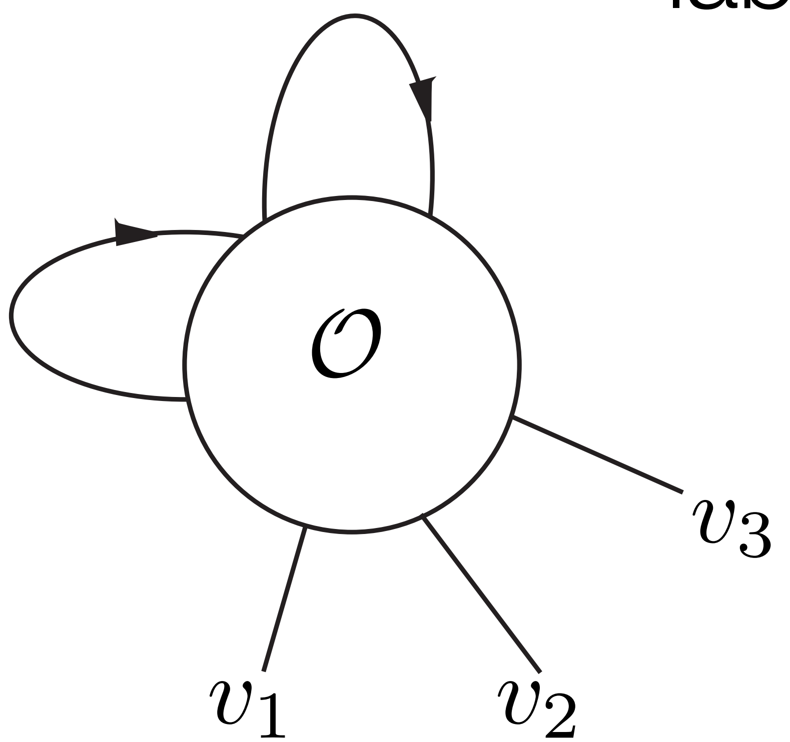


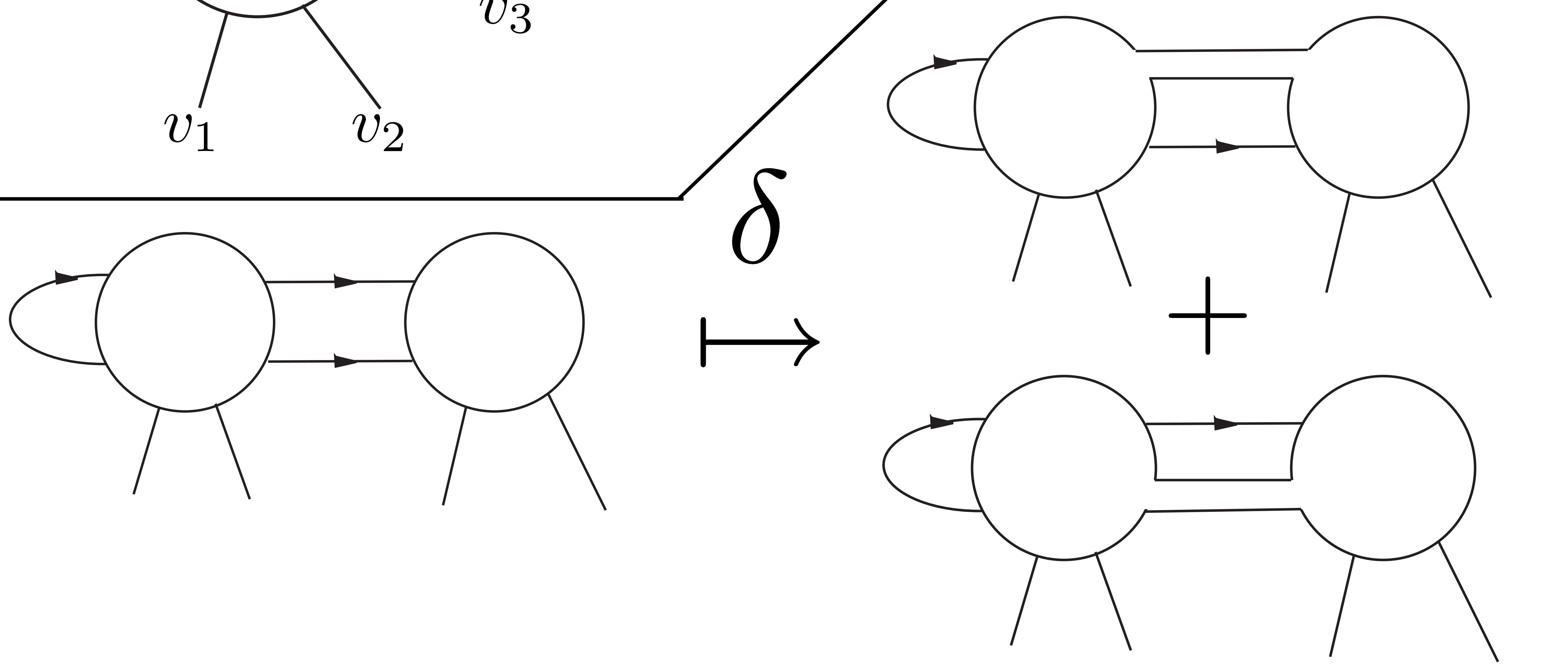
$$\xrightarrow{\;Tr\;} \sum_M \langle v_{i_1}, v_{j_1}\rangle \cdots \langle v_{i_k}, v_{j_k}\rangle$$



Theorem: $(\ell_\infty^+)_{ab} \overset{Tr_*}{\hookrightarrow} H_1(\mathcal{HG})$

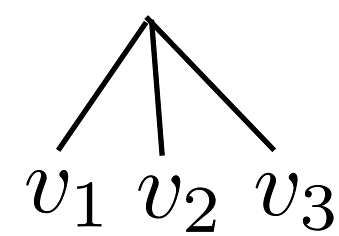
where $Im(Tr_*)$ contains, for example, hairy graphs labeled by $V^+ \subset V$.



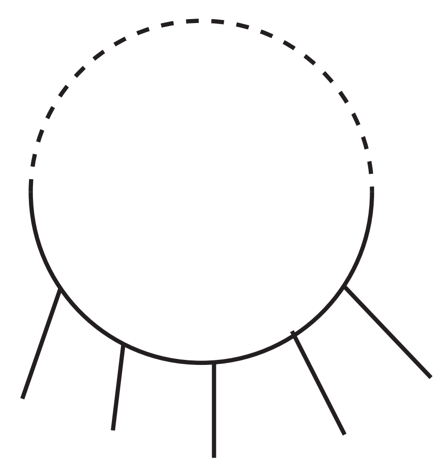


$H_1(\mathcal{HG})$ is graded by loop degree.

$$H_1^0(\mathcal{HG}) \cong \Lambda^3 V$$

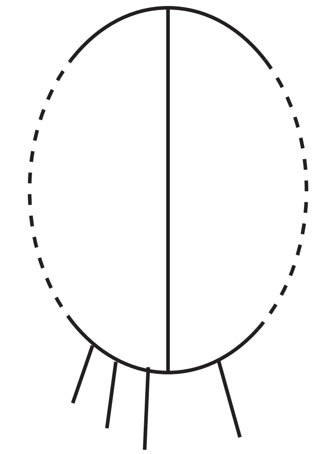


$$H_1^1(\mathcal{HG}) \cong \bigoplus_{k=1}^{\infty} S^{2k+1} V$$



Morita

New: $H_1^2(\mathcal{HG}) \cong \bigoplus_{k>\ell\geq 0} (F_{(k,\ell)})^{\oplus \lambda_{k,\ell}}$



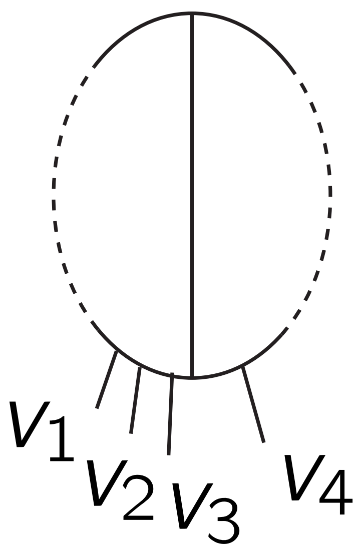
$F_{(k,\ell)}$ = irrep of GL(V)

s_n is the dimension of the space of weight n cuspidal modular forms for $SL(2,\mathbb{Z})$.

$$\lambda_{k,\ell} = \begin{cases} s_{k-\ell+2} & \text{if } \ell \text{ is even.} \\ s_{k-\ell+2}+1 & \text{if } \ell \text{ is odd.} \end{cases}$$

New: $H_1^3(\mathcal{HG}) \neq 0$

Example:



$$\neq 0 \in H^2_1(\mathcal{HG})$$

$v_i \in V^+ \Rightarrow$

this is in im(Tr), so represents a nonzero element of $(\ell^+_\infty)_{ab}$.

$\langle v_3, v_4 \rangle \neq 0 \Rightarrow$

this is not in im(Tr).

Proof of $H_1^2(\mathcal{HG}) \cong \bigoplus_{k>\ell\geq 0} (F_{(k,\ell)})^{\bigoplus \lambda_{k,\ell}}$

Step 1: $H_1^r(\mathcal{HG}) \cong H^{2r-3}(Out(F_r);\mathcal{P}(V^{\oplus r}))$

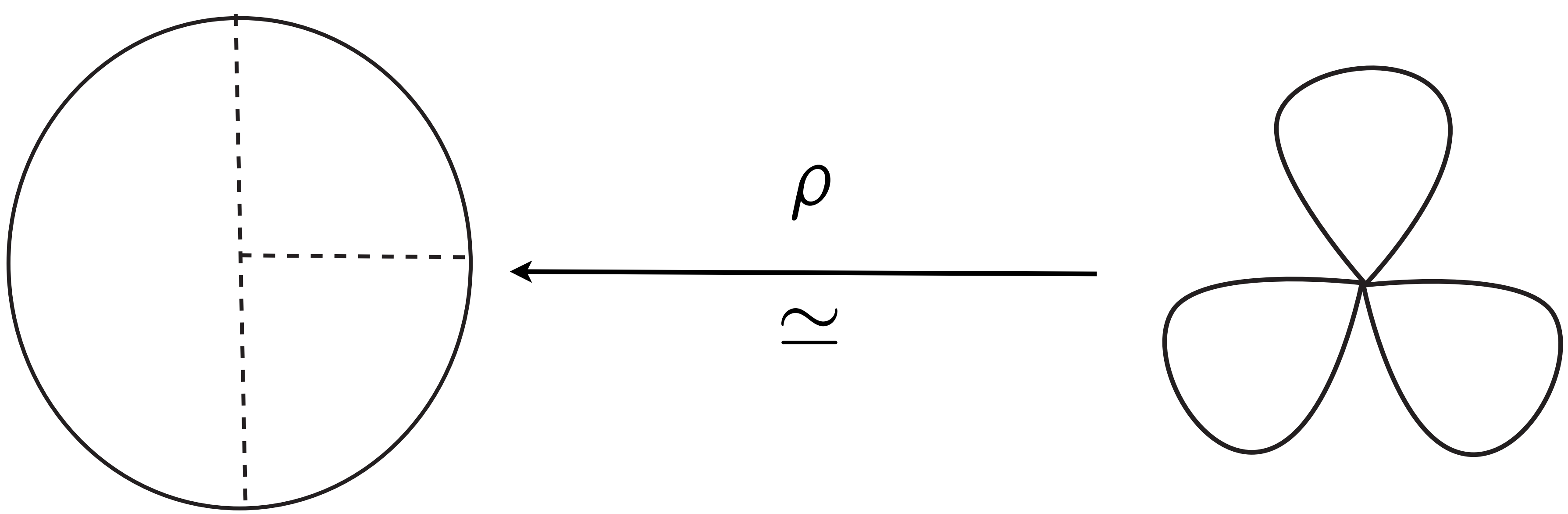
Step 2:

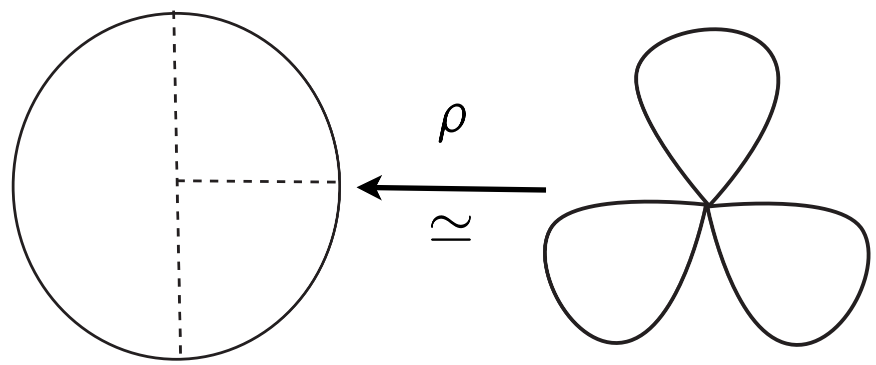
$$H^1(Out(F_2);\mathcal{P}(V \oplus V)) = H^1(GL(2,\mathbb{Z});\mathcal{P}(V \oplus V))$$

Use existing results (Eichler-Shimura).

Proof of Step 1:

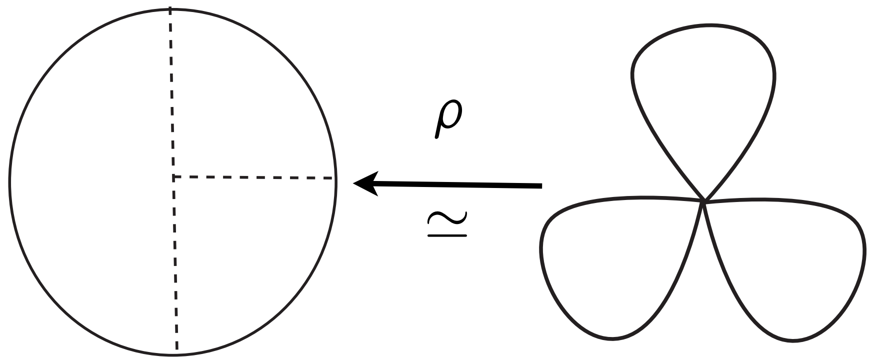
Spine of Outer Space





$$\otimes \quad \Phi \in \tilde{C}^{2r-3}(Out(F_r), \mathcal{P}(V^{\oplus r}))$$

$$\Phi \in \mathcal{P}(V^{\oplus r}) = \mathcal{P}(V \otimes H^1(R_r, \mathbb{C})) \overset{\rho^*}{\cong} \mathcal{P}(V \otimes H^1(G, \mathbb{C}))$$



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$$\mathcal{P}(V \otimes H^1(G, \mathbb{C})) \leftrightarrow \text{hairy graphs}$$

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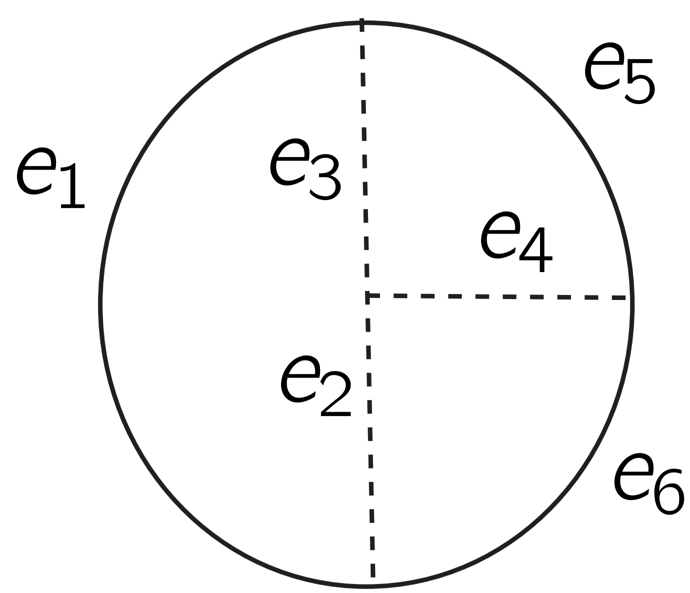
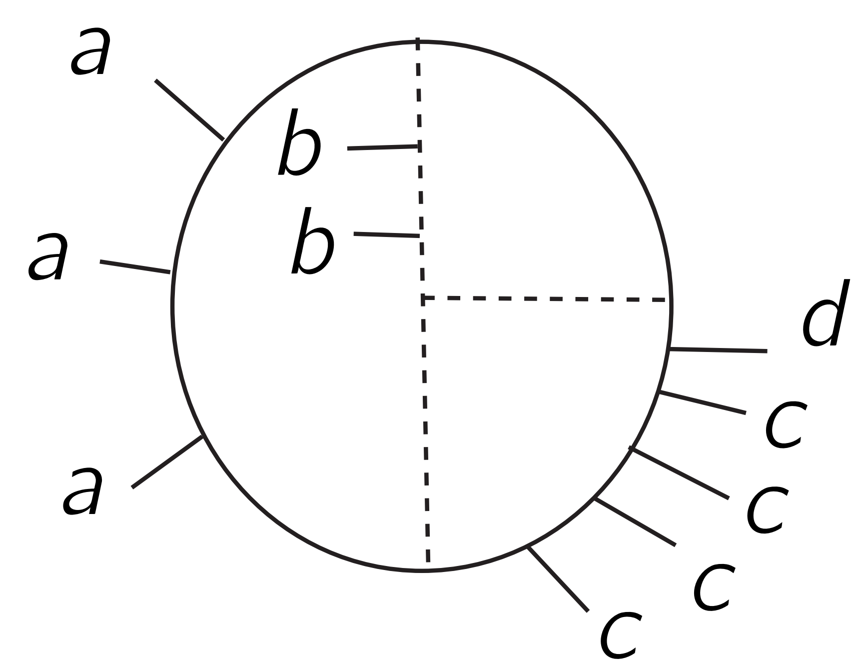
$$(e_1 \otimes a)^3 (e_3 \otimes b)^2 (e_6 \otimes c)^4 (e_6 \otimes d)$$





Modulo the action of $Out(F_r)$ we are left with hairy graphs up to graph isomorphism.

One verifies that in this top degree, the hairy graph boundary operator corresponds to the boundary operator for the spine (with local coefficients.)

$\square$

Step 2: $H^1(Out(F_2), \mathcal{P}(V \otimes \mathbb{C}^2)) =?$

Detour: modular forms.

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$$\begin{tikzcd} \mathbb{H} \arrow[r, "f"] \arrow[d, "q(z) = e^{2\pi i z}"'] & \mathbb{C} \\ \mathbb{C} \setminus \{0\} \arrow[ur, dotted, "f_\infty"'] & \end{tikzcd}$$

f(z)=f(z+1) so this ‘q-expansion’ exists.

f_∞ meromorphic on $\mathbb{C} \Rightarrow$

f is a *modular form* of weight k.

$f_\infty(0) = 0 \Rightarrow$

f is *cuspidal*.

Example: Eisenstein Series

$$G_k(z) = \sum_{(m,n)\neq(0,0)} \frac{1}{(mz+n)^k} \quad k > 2$$

Theorem: The complex vector space of modular forms is isomorphic to the polynomial ring $\mathbb{C}[G_4, G_6]$.

Exercise:

$$\dim M_k = \begin{cases} \lfloor \frac{k}{12} \rfloor & \text{if } k \equiv 2 \mod 12 \\ \lfloor \frac{k}{12} \rfloor + 1 & \text{if } k \not\equiv 2 \mod 12 \end{cases}$$

$$M_k \cong M_k^0 \oplus \mathbb{C}$$

Eichler-Shimura isomorphism

Let f be a cusp form of weight k.

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$$\omega(f) = \begin{bmatrix} f(z)z^{k-2}\,dz \\ f(z)z^{k-3}\,dz \\ \vdots \\ f(z)z^{0}\,dz \end{bmatrix}$$

Eichler-Shimura isomorphism

$$H^1(SL(2,\mathbb{Z}); H_{k-2}) \cong M_k^0 \oplus \overline{M_k^0} \oplus E_k$$

Let s_n be the dimension of the space of weight n cuspidal modular forms for $SL(2,\mathbb{Z})$.

Let $F_{(k,\ell)}$ be the irreducible representation of GL(V) associated to the partition (k,ℓ), $k \geq \ell$.

Theorem: $H^1(Out(F_2);\mathcal{P}(V\otimes\mathbb{C}^2)) \cong \bigoplus_{k>\ell\geq 0} (F_{(k,\ell)})^{\oplus\lambda_{k,\ell}}$

where $\lambda_{k,\ell} = \begin{cases} s_{k-\ell+2} & \text{if } \ell \text{ is even.} \\ s_{k-\ell+2}+1 & \text{if } \ell \text{ is odd.} \end{cases}$

Proof:

$$Out(F_2) = GL(2, \mathbb{Z})$$

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irreps for V and $\mathbb{C}^2$.

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$$\mathcal{P}[V \otimes \mathbb{C}^2] \cong \bigoplus_{m \geq n} H_{m-n} \otimes F_{(m,n)}$$

$$H^1(GL(2,\mathbb{Z}); \mathcal{P}[V \otimes \mathbb{C}^2]) = \bigoplus_{m \geq n} H^1(GL(2,\mathbb{Z}); H_{m-n}) \otimes F_{(m,n)}$$

$$H^1(GL(2,\mathbb{Z}),H_k) \cong H^1(SL(2,\mathbb{Z}),H_k)^{\mathbb{Z}_2}$$

$$H^1(SL(2,\mathbb{Z});H_k) \cong M^0(k+2) \oplus \overline{M^0(k+2)} \oplus E_{k+2}$$

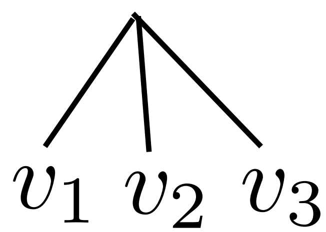
$$H^1(GL(2,\mathbb{Z});\mathcal{P}[V \otimes \mathbb{C}^2]) = \bigoplus_{m \geq n} H^1(GL(2,\mathbb{Z});H_{m-n}) \otimes F_{(m,n)}$$

$$H^1(SL(2,\mathbb{Z});H_{m-n})^{\mathbb{Z}_2} \cong \begin{cases} M^0(m-n+1) & \text{if } n \text{ is even.} \\ M^0(m-n+2) \oplus E_{m-n+2} & \text{if } n \text{ is odd.} \end{cases}$$

$E_{m-n+2} \cong \mathbb{C}$ unless $m = n$ $\qquad$ $E_2 = 0$

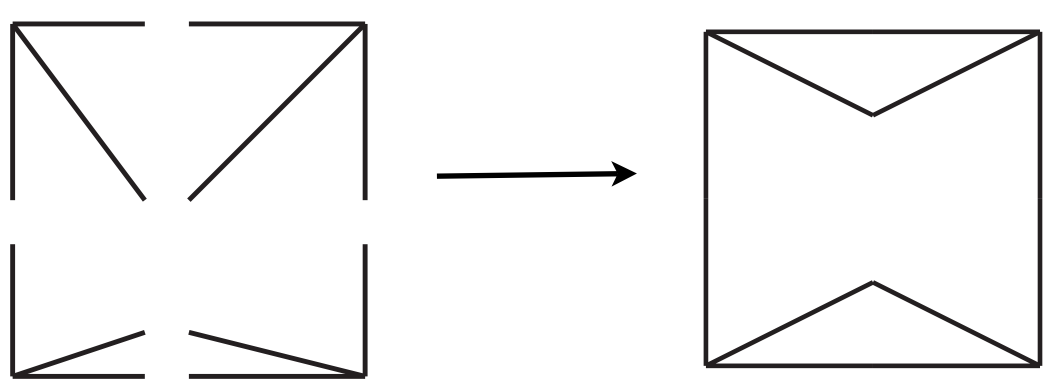
$\square$

Fitting pieces of the abelianization together

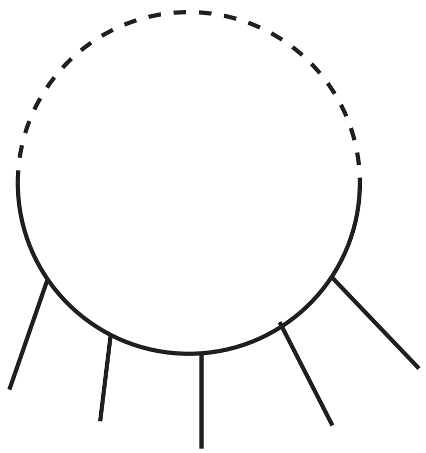


$\Lambda^3 V$ can combine with themselves to detect

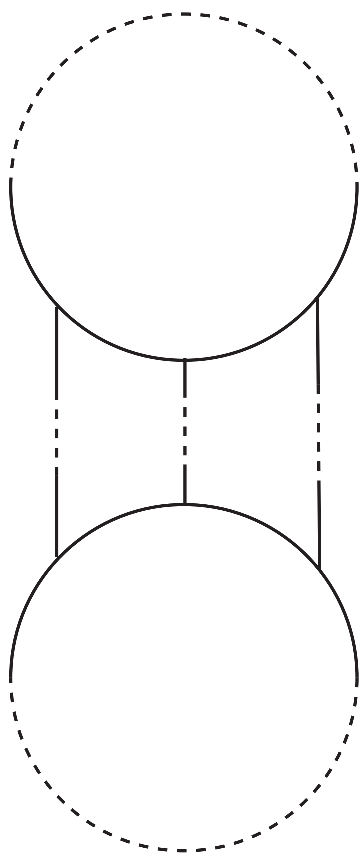
$$H_0(Out(F_r);\mathbb{Q}) \cong \mathbb{Q}$$



$$H_0(Out(F_3);\mathbb{Q})$$

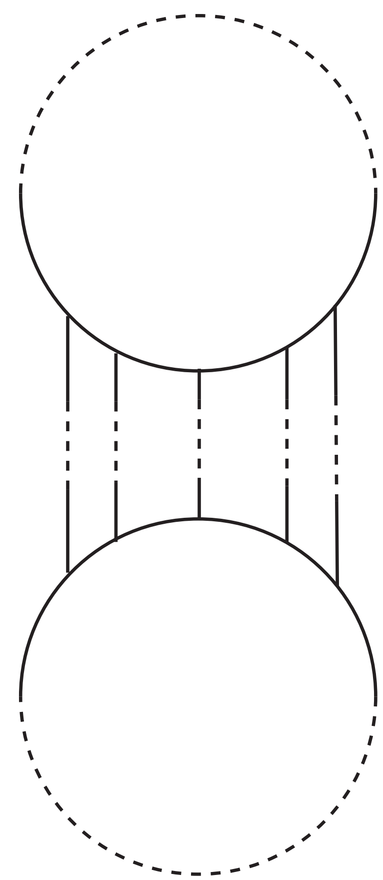


$\bigoplus_{k=1}^{\infty} S^{2k+1}V$ can combine with themselves to create generalized Morita classes.



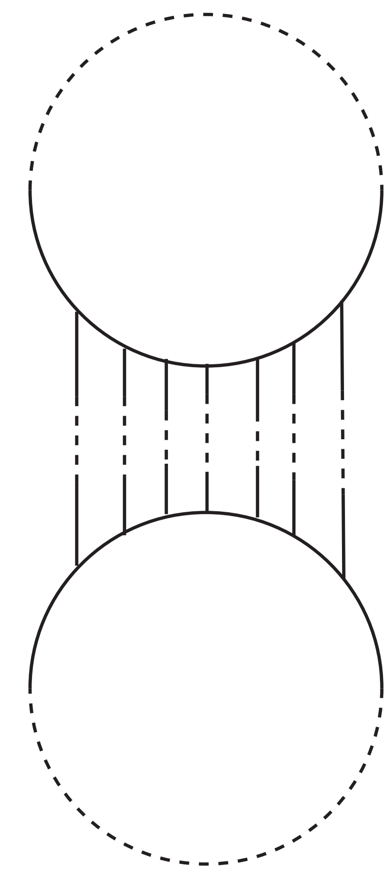
$H_4(Out(F_4);\mathbb{Q})$

(Vogtmann)



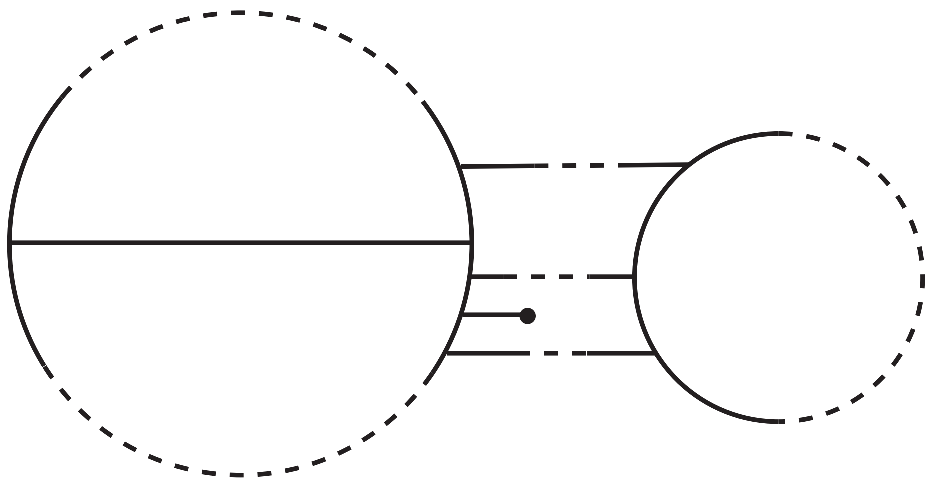
$H_8(Out(F_6);\mathbb{Q})$

(Conant,Vogtmann, Ohashi)



$H_{12}(Out(F_8);\mathbb{Q})$

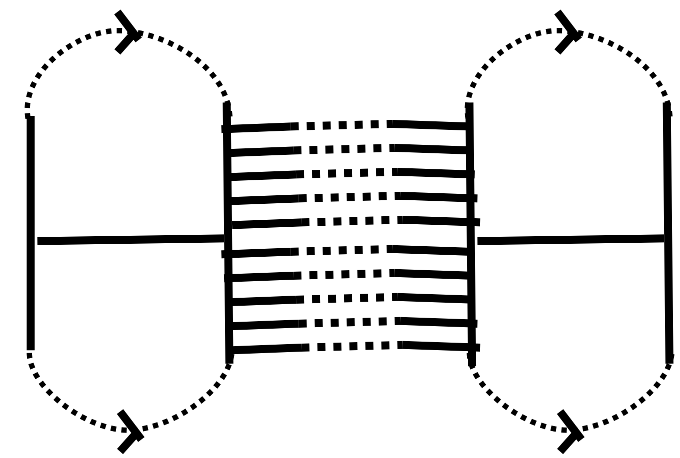
(Gray)



$H_7(Aut(F_5);\mathbb{Q})$

$H_{11}(Aut(F_7);\mathbb{Q})$

(CKV, Gerlits)



$H_{22}(Out(F_{13});\mathbb{Q})$

nonzero??

Further Directions

- Extend the 2-loop calculation to 3-loops and beyond. To generalize our argument, we need the cohomology of SL(n,Z) with coefficients in an irreducible representation (doable) as well as the cohomology of IA_n as a GL module, which is quite hard.

- Show that classes produced from gluing together graphs in the abelianization give rise, in some large number of case, to nontrivial homology classes. Current methods require computer computations. The next Morita class is probably within reach, but essentially a new method will be needed for the general case.

- All known classes for Aut and Out arise from this abelianization construction. Is this true in general?