Automorphisms of Right-Angled Artin Groups

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Joint work with

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Notation:

Γ = finite, simplicial graph

$V = \{v_1, \ldots, v_n\}$ = vertex set

$A_\Gamma = \langle V \mid v_i v_j = v_j v_i$, iff v_i, v_j are adjacent in $\Gamma \rangle$

= right-angled Artin group (RAAG)

dim A_Γ = size of maximal clique in Γ

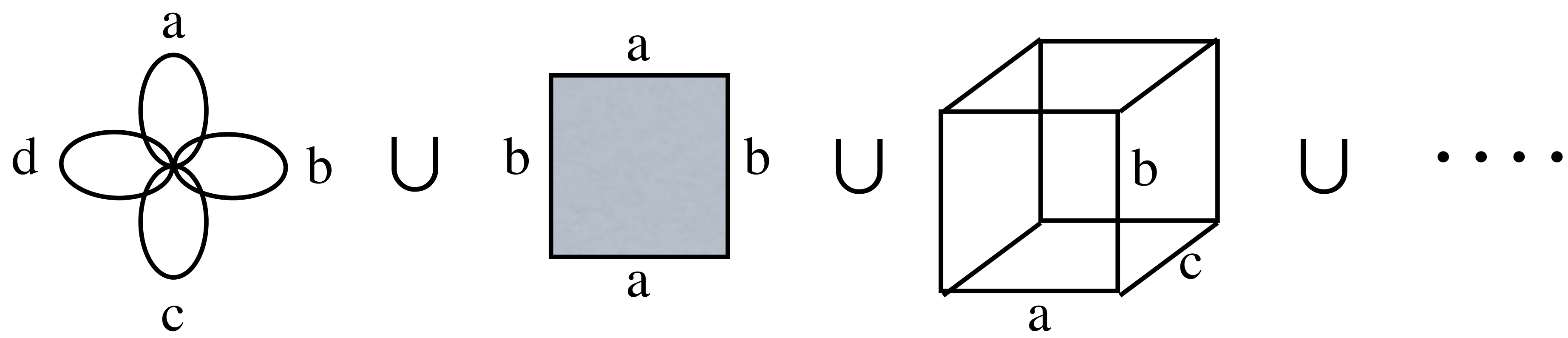
= rank of maximal abelian subgroup of A_Γ

dim = 1 $\Rightarrow$ A_Γ = free group

dim = n $\Rightarrow$ A_Γ = free abelain group

$K(A_\Gamma,1)$-space: Salvetti complex for A_Γ

S_Γ = Rose $\cup$ (k-torus for each k-clique in Γ)



S_Γ is a locally CAT(0) cube complex with fundamental group A_Γ.

$A_\Gamma \curvearrowright \widetilde{S}_\Gamma$ = CAT(0) cube complex, dim $\widetilde{S}_\Gamma$ =dim A_Γ

Right-angled Artin groups

- have nice geometry
- contain interesting subgroups
- interpolate between free groups and free abelian groups

They provide a context to understand the relation between

$Out(F_n)$ Linear groups MCG

$$Out(F_n) \overset{Out(A_\Gamma)}{\longleftrightarrow} GL_n(Z)$$

$$Sp_{2g}(Z) \overset{Out(A_\Gamma,\omega)}{\longleftrightarrow} MCG(S_g)$$ (M. Day)

Many properties are known to hold for

$Out(F_n)$ and $GL_n(Z)$

Which of these properties hold

for *all* $Out(A_\Gamma)$?

Some results:

- $Out(A_\Gamma)$ is virtually torsion-free, finite vcd
- Bounds on vcd
- $Out(A_\Gamma)$ is residually finite (proved independently by Minasyan)
- $Out(A_\Gamma)$ satisfies the Tits alternative (if Γ homogeneous)

Some techniques of proof

Definition: Let $\Theta \subset \Gamma$ be a full subgraph. Say Θ is *characteristic* if every automorphism of A_Γ preserves A_Θ up to conjugacy (and graph symmetry).

Say $\Theta \subset \Gamma$ is characteristic. Then

$$A_\Theta \hookrightarrow A_\Gamma \twoheadrightarrow A_{\Gamma\backslash\Theta} \cong A_\Gamma / \langle\langle A_\Theta \rangle\rangle$$

induces restriction and exclusion homomorphisms:

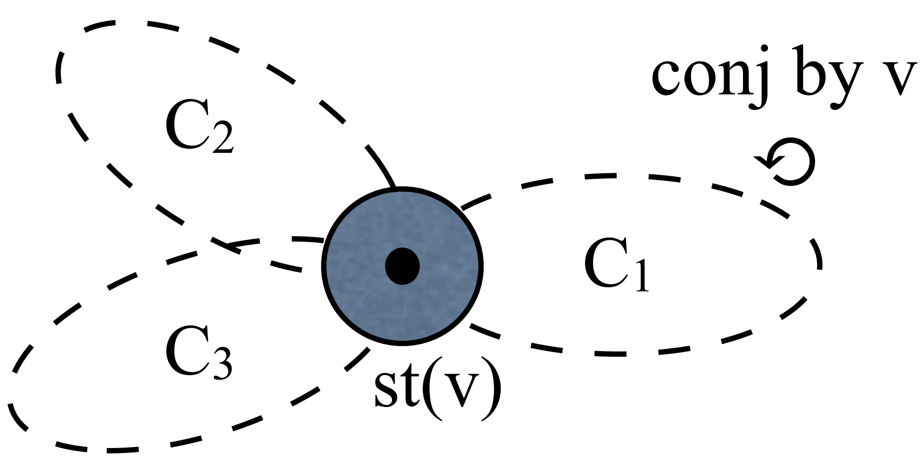
$$\mathrm{Out}\,(A_\Theta) \xleftarrow{R_\Theta} \mathrm{Out}\,(A_\Gamma) \xrightarrow{E_\Theta} \mathrm{Out}\,(A_{\Gamma\backslash\Theta})$$

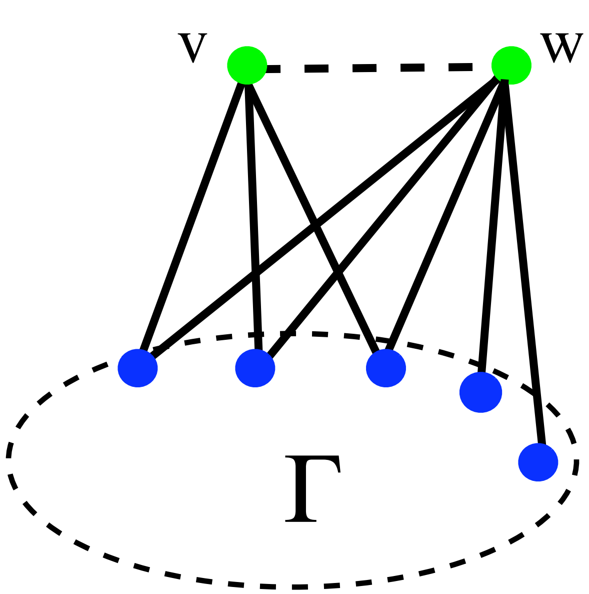
Main idea: use these to reduce questions about $\mathrm{Out}(A_\Gamma)$ to questions about some smaller $\mathrm{Out}(A_\Theta)$ and use induction.

How can we find characteristic subgraphs?

Servatius ('89), Laurence ('95): $Out(A_\Gamma)$ has a finite generating set consisting of:

- Graph symmetries: $\Gamma \to \Gamma$
- Inversions: $v \to v^{-1}$
- Partial conjugations: conjugate a connected component of $\Gamma \backslash st(v)$ by v.
- Transvections: $v \to vw$, providing $lk(v) \subset st(w)$



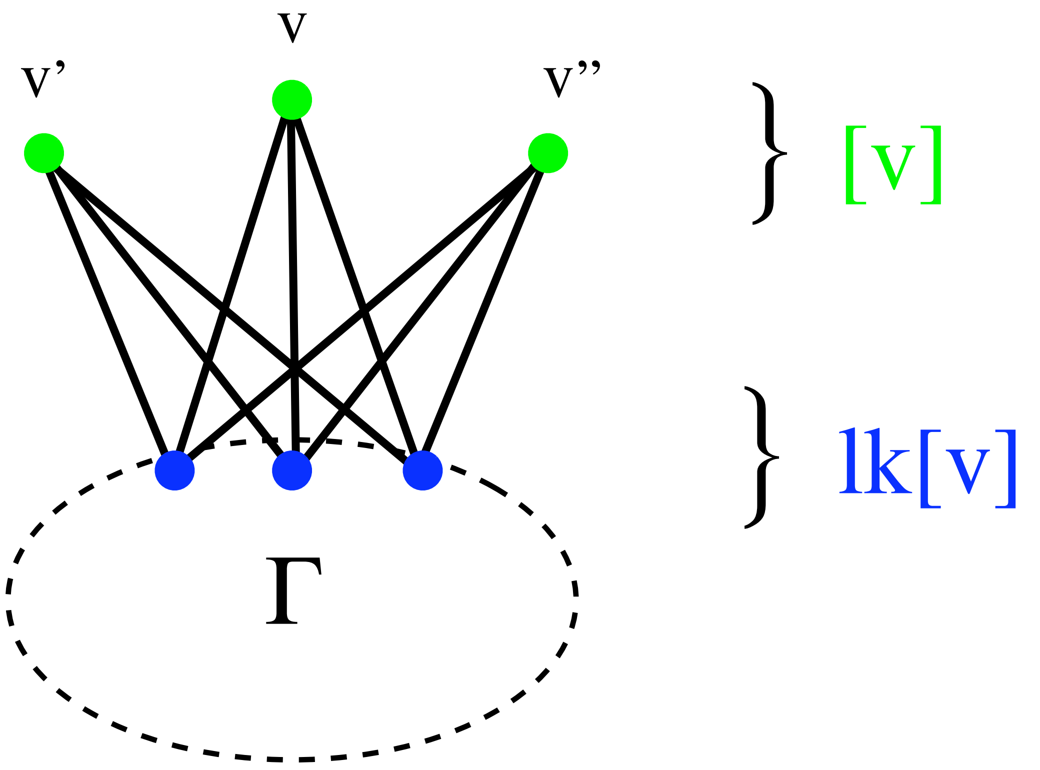


Define $Out^0(A_\Gamma)$ = subgroup generated by inversions, partial conjugations, transvections

Define a partial ordering on vertices of Γ

$v \leq w$ if $lk(v) \subset st(w)$

$v \sim w$ if $v \leq w$ and $w \leq v$



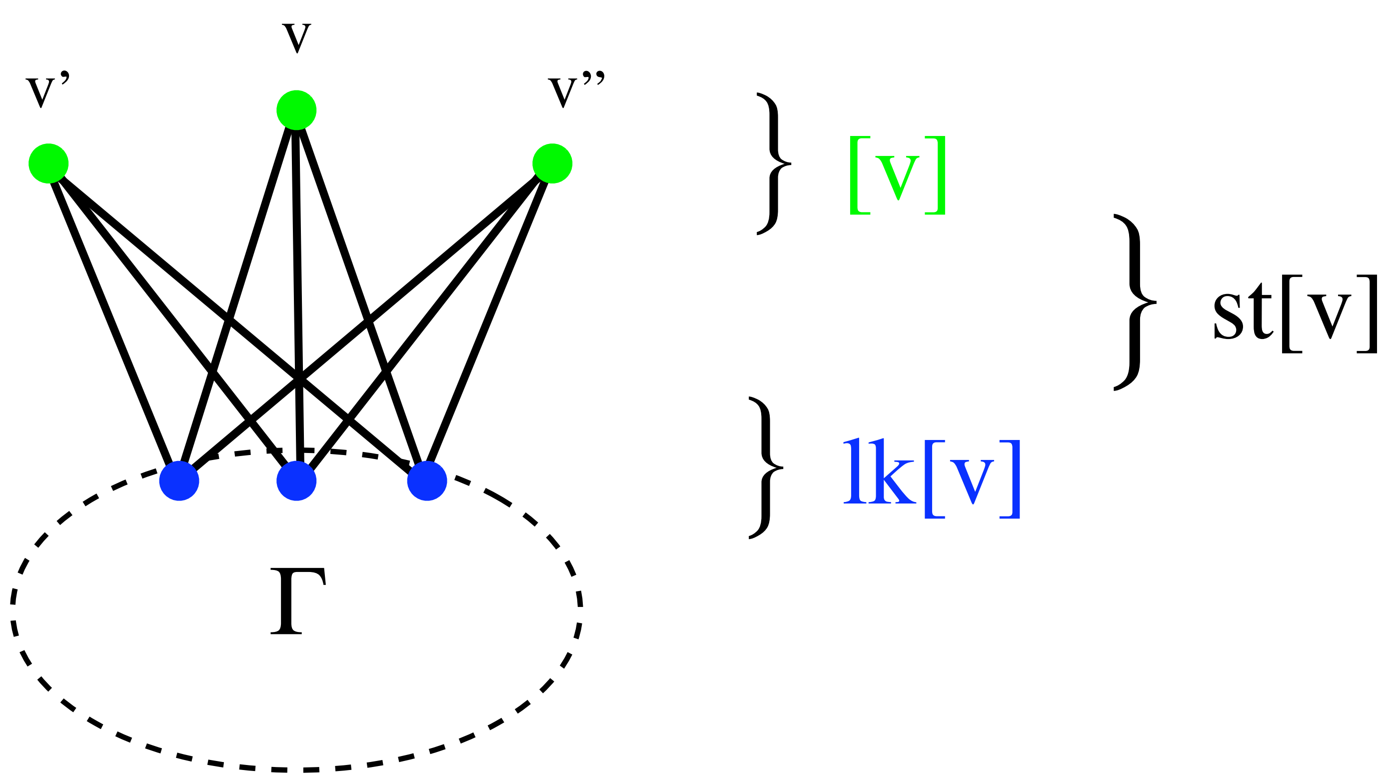
Let $[v]$ = equivalence class of v

$st[v] = \bigcup_{w \sim v} st(w)$

$lk[v] = st[v] \setminus [v]$

If [v] is maximal, then [v] and st[v] are characteristic!

Proof: check that each of the Servatius-Laurence generators preserves $A_{[v]}$ and $A_{st[v]}$ up to conjugacy.



So if [v] is maximal, we have a homomorphism

$$P_{[v]}: Out^0(A_\Gamma) \xrightarrow{R} Out^0(A_{st[v]}) \xrightarrow{E} Out^0(A_{lk[v]})$$

Key Lemma: If Γ is connected, then the kernel K of

$$1 \rightarrow K \rightarrow Out^0(A_\Gamma) \xrightarrow{p} \prod Out^0(A_{lk[v]})$$

is a finitely generated free abelian group. (We give explicit generating set for K.)

Key Lemma: If Γ is connected, then the kernel K of

$$1 \to K \to Out^0(A_\Gamma) \to \prod Out^0(A_{lk[v]})$$

is a finitely generated free abelian group.

Theorem: (C-Crisp-Vogtmann, C-Vogtmann) For all right-angled Artin groups A_Γ, $Out(A_\Gamma)$ is virtually torsion-free and has finite virtual cohomological dimension (vcd).

Proof: Induction on dim A_Γ.
dim $A_\Gamma = 1$ means dim A_Γ = free group. True by Culler-Vogtmann.
Say dim $A_\Gamma > 1$. Note that dim $A_{lk[v]} <$ dim A_Γ for all [v].
So by induction, $Out(A_\Gamma)$ is virtually torsion-free and has finite vcd, *providing* Γ *is connected*.
If Γ is disconnected, A_Γ is a free product and can use results of Guirardel-Levitt on Out(free products).

Also get bounds on the vcd.

Theorem: (C-Bux-Vogtmann) If Γ is a tree, then

$$\mathrm{vcd}(\mathrm{Out}(A_\Gamma)) = e + 2l - 3$$

where e = # edges and l = # leaves.

Proof: In this case $A_{lk[v]}$ is free. We identify of the image of $P\colon \mathrm{Out}(A_\Gamma) \to \prod \mathrm{Out}(A_{lk[v]})$ and compute its vcd by finding an invariant subspace of outer space.

Theorem: (C-Vogtmann) For all A_Γ, $\mathrm{Out}(A_\Gamma)$ is residually finite.

Proof: Use Key Lemma as before,

$$1 \to K \to \mathrm{Out}^0(A_\Gamma) \xrightarrow{P} \prod \mathrm{Out}^0(A_{lk[v]})$$

to show that its true for connected Γ. Use results of Minasyan-Osin for free products.

Tits Alternative

A group G satisfies the Tits Alternative if every subgroup of G is either virtually solvable or contains F_2.

A group G satisfies the Strong Tits Alternative if every subgroup of G is either virtually abelian or contains F_2.

A_Γ = free group, $Out(A_\Gamma)$ satisfies the Strong Tits Alternative

A_Γ = free abelian, $Out(A_\Gamma)=Gl(n, Z)$ satisfies the Tits Alternative and has non-abelian solvable subgroups.

What about the Tits Alternative for other $Out(A_\Gamma)$?

Try to prove Tits Alternative for $\mathrm{Out}(A_\Gamma)$
by induction as above.
Problem: cant get from connected $\Rightarrow$ disconnected Γ

Question: If $G = G_1 * \ldots * G_k$ and $\mathrm{Out}(G_i)$ satisfies the Tits Alternative for all i, does the same hold for Out(G)?

Definition: Γ is homogeneous of dim 1 if Γ is discrete.
Γ is homogeneous of dim n if Γ is connected and lk(v) is homogeneous of dim n-1 for all v.

Example: The 1-skeleton of any triangulation of a n-manifold is homogeneous of dimesnion n.

Theorem: (C-Vogtmann) Assume Γ is homogeneous of dim n. Then

1. $Out(A_\Gamma)$ satisfies the Tits Alternative.
2. The derived length of every solvable subgroup is $\leq n$.
3. $\widetilde{Out}(A_\Gamma)$ satisfies the Strong Tits Alternative.

(where $\widetilde{Out}(A_\Gamma)$ is the subgroup generated by all of the Servatius-Laurence generators, *except* adjacent transvections.)

Corollary: If Γ is a connected graph with no triangles and no leaves, then $Out(A_\Gamma) = \widetilde{Out}(A_\Gamma)$ satisfies the Strong Tits Alternative.

Proof: (1) and (2) follow from key lemma and induction.
To prove (3), must show virtually solvable $\Rightarrow$ virtually abelian.
Conner, Gersten-Short: true if every ∞-order element has positive translation length, $\tau(g) = \lim_{k\to\infty} \frac{\| g^k \|}{k} > 0$.

Work in Progress

Find an “outer space” for $Out(A_\Gamma)$

Outer space for F_n, $\mathcal{CV}(F_n)$:

(1) equiv classes of marked metric graphs

$$\text{Rose} \xrightarrow{\simeq} \Theta$$

(2) minimal, free actions of F_n on a tree

What is the analogue for $Out(A_\Gamma)$?

Example: $A_\Gamma = F_n \times F_m \curvearrowright$ tree $\times$ tree

so natural choice for outer space would be

$$\mathcal{CV}(A_\Gamma) = \{\text{minimal, free actions of } A_\Gamma \text{ on tree} \times \text{tree}\}$$

More generally, if dim $A_\Gamma = 2$, then for every $[v]$,

$$A_{st[v]} = A_{[v]} \times A_{lk[v]} = \text{free} \times \text{free}$$

C-Crisp-Vogtmann: For dim $A_\Gamma = 2$, we construct an "outer space"

$$\mathcal{CV}_1(A_\Gamma) = \{ (A_{[v]} \times A_{lk[v]} \curvearrowright \text{tree} \times \text{tree}), \text{ compatibility data}\}$$

Theorem: For dim $A_\Gamma = 2$, $\mathcal{CV}_1(A_\Gamma)$ is contractible and has a proper action of $\mathrm{Out}(A_\Gamma)$.

However, $\mathcal{CV}_1(A_\Gamma)$ is very big and somewhat awkward.

Back to our example:

$A_\Gamma = F_n \times F_m \curvearrowright$ tree $\times$ tree = CAT(0) rectangle complex

so a more natural choice for outer space might be

$$\mathcal{CV}_2(A_\Gamma) = \{\text{minimal, free actions of } A_\Gamma \text{ on a CAT(0) rectangle complex}\}$$

$$= \{\text{marked, locally CAT(0) rectangle complexes, } S_\Gamma \xrightarrow{\simeq} X\}$$

Conjecture: $\mathcal{CV}_2(A_\Gamma)$ (or some nice invariant subspace) is contractible.

Culler-Morgan: A minimal, semi-simple action

$F_n \curvearrowright$ tree is uniquely determined (up to equivariant isometry) by its length function .

$$l(g) = \inf \{d(x,gx) \mid x \in X\}$$

This gives an embedding

$$\mathcal{CV}(F_n) \hookrightarrow \mathbb{P}^\infty = \mathbb{P}^{C(F_n)}$$

whose closure $\overline{\mathcal{CV}}(F_n)$ is compact.

Theorem: (C-Margolis) For dim $A_\Gamma = 2$, a minimal, free action of A_Γ on a 2-dim'l CAT(0) rectangle complex is determined (up to equivariant isometry) by its length function. Thus,

$$\mathcal{CV}_2(A_\Gamma) \hookrightarrow \mathbb{P}^\infty = \mathbb{P}^{C(A_\Gamma)}$$

Question: Is $\overline{\mathcal{CV}_2}(A_\Gamma)$ compact?

$F_n \curvearrowright T$ is minimal if T is the union of the axis of elements of F_n. (axis(g)={x | d(x,gx) is minimal})

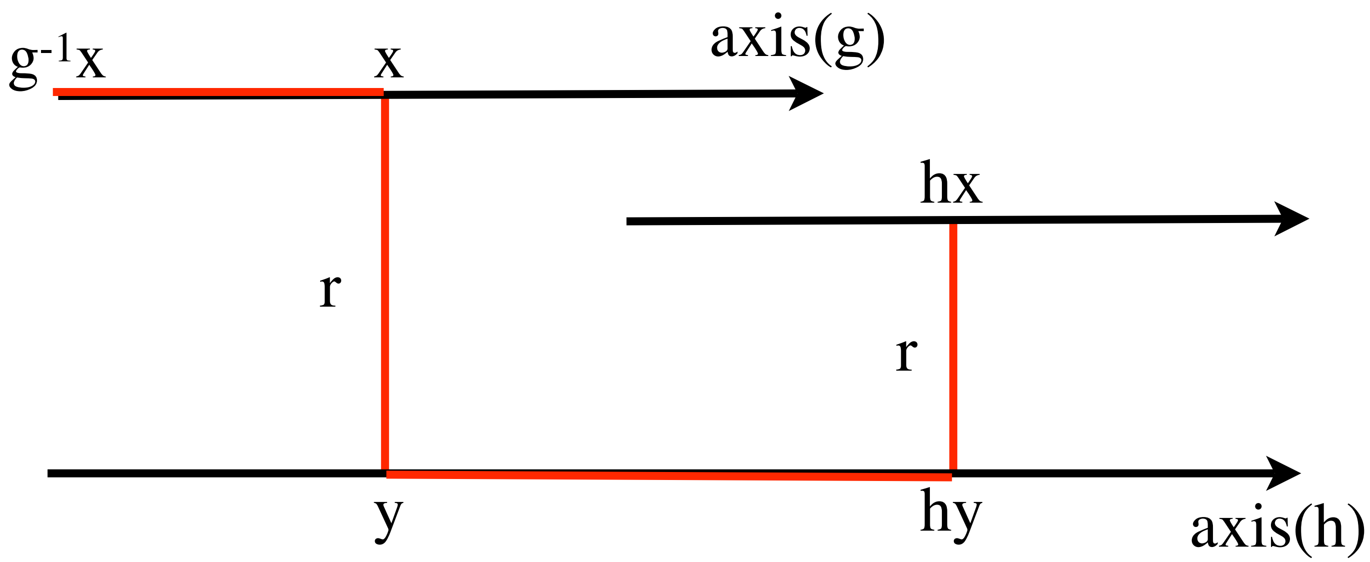
Def: $A_\Gamma \curvearrowright X$ is minimal if X is the union of the minsets of rank 2 abelian subgroups.

(If dim X=2, this implies $X = \cup$ 2-flats)

Proof of Theorem: Show length function determines

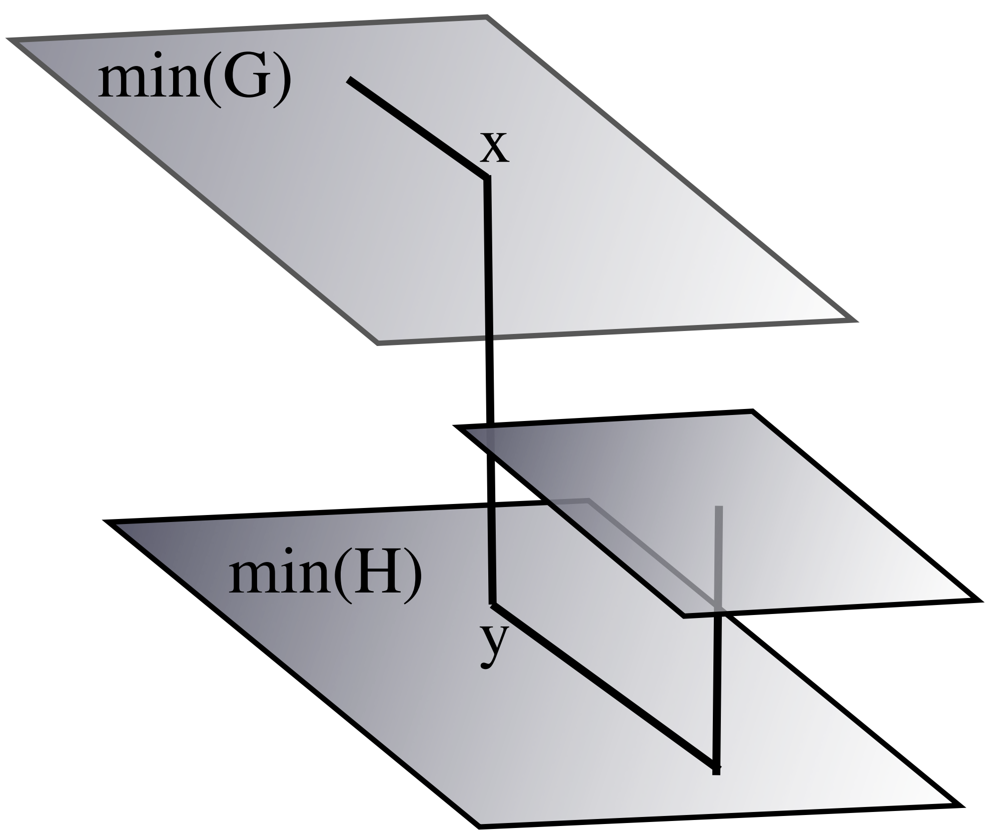
- distance between any two such flats
- shape of intersection of any two flats

$F_n \curvearrowright$ tree: Distance between non-intersecting axes



$l(hg) = l(h) + l(g) + 2r$

$A_\Gamma \curvearrowright X$: Distance between non-intersecting flats:



May not be geodesic, so
$l(hg) \leq l(h) + l(g) + 2r$

We show that
$2r = \sup \{l(hg) - l(h) - l(g)\}$