Complementary Series of Split Real Groups

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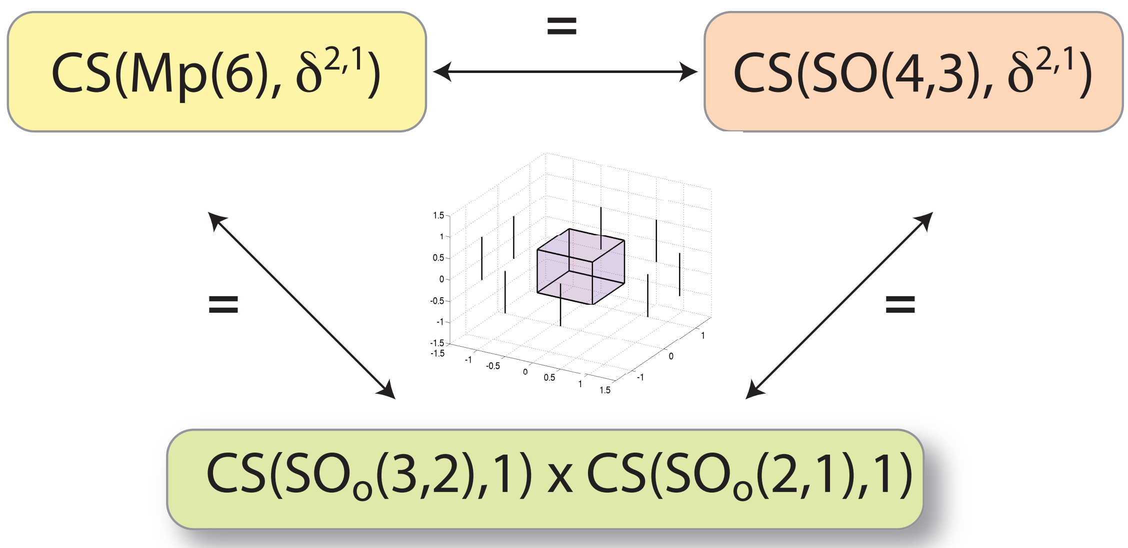
joint with Annegret Paul and Susana Salamanca-Riba

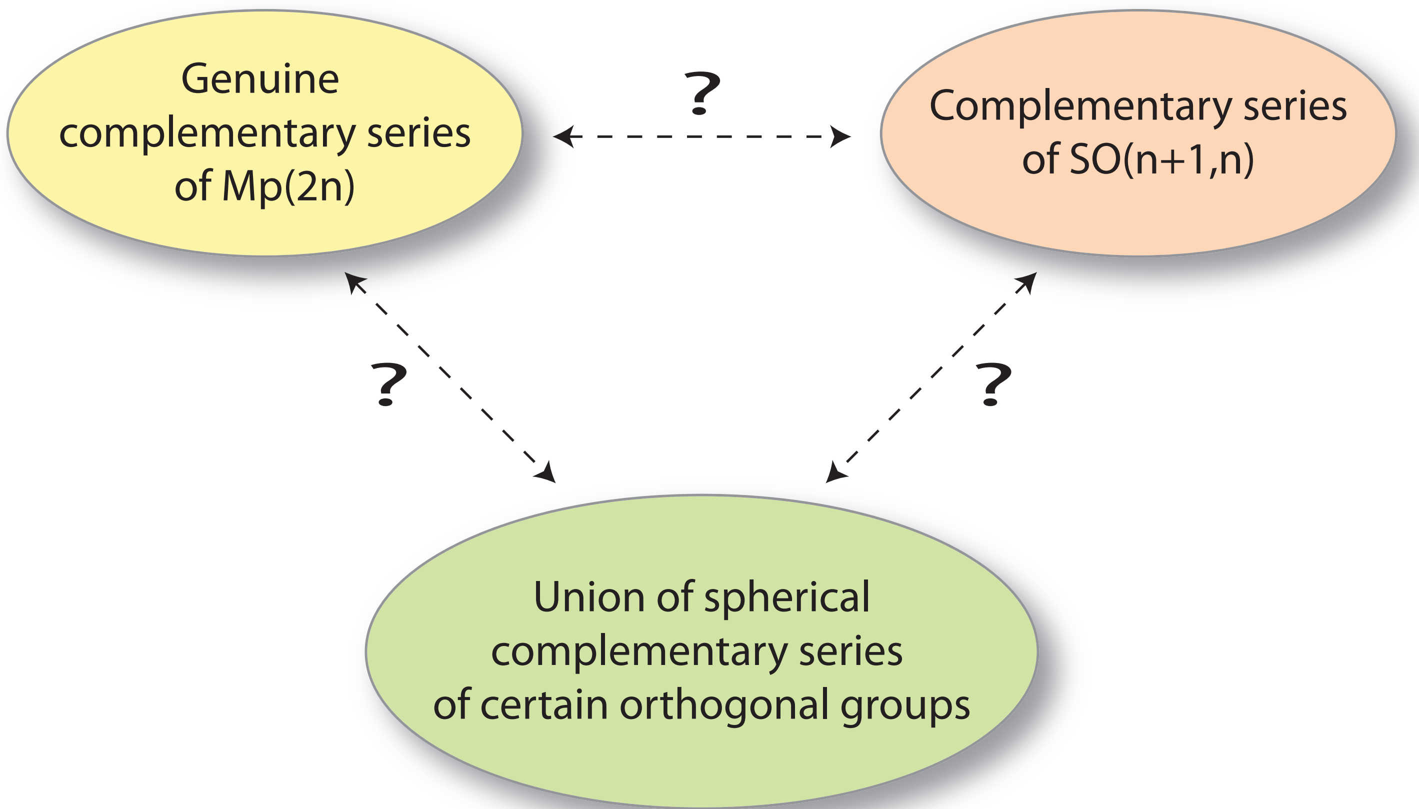
(some of the techniques used are joint work with D. Barbasch)



Salt Lake City, July 2009

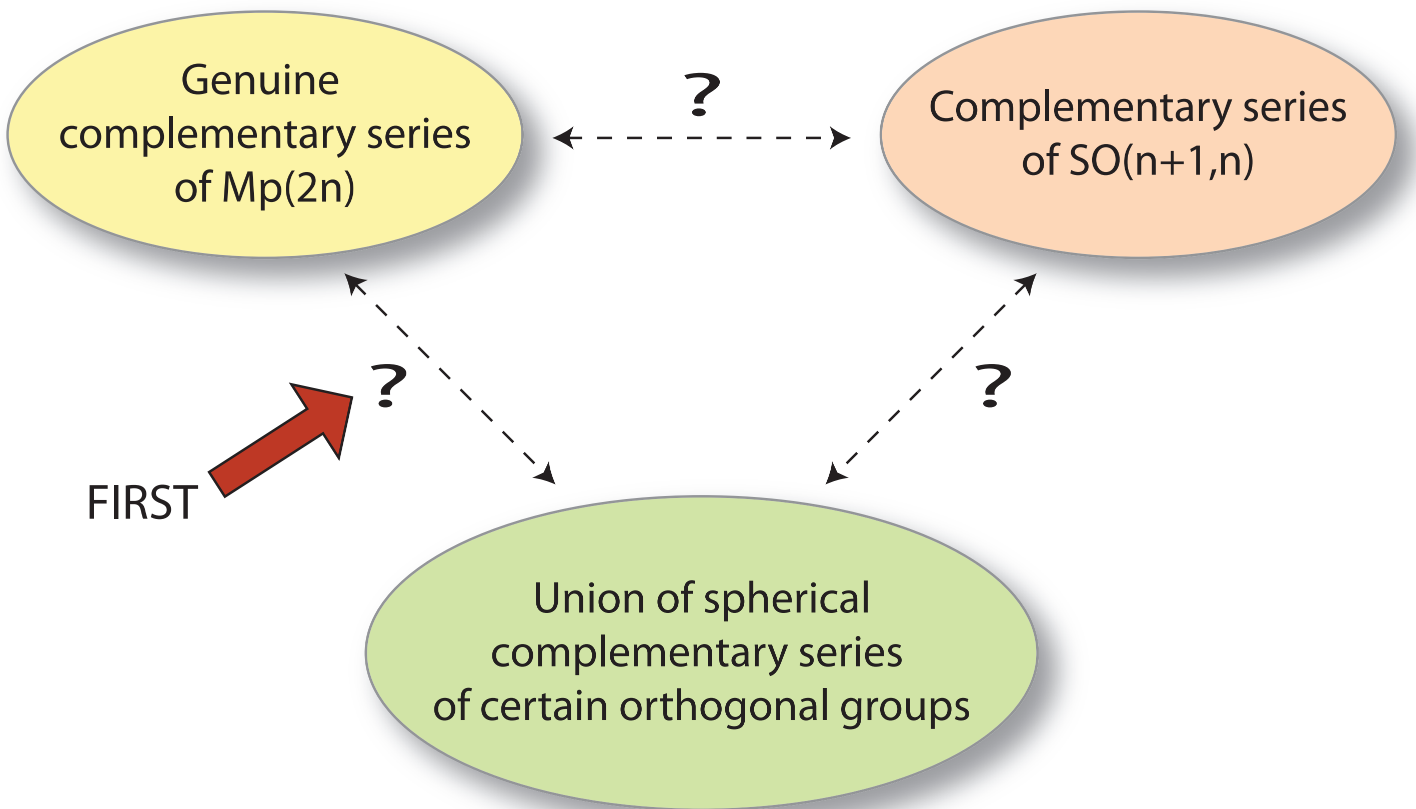
Introduction

Aim Discuss the unitarity of minimal principal series of $Mp(2n)$ and $SO(n+1,n)$.



PART 1

Genuine Complementary Series of $Mp(2n)$



NOTATION

- $G := Mp(2n)$ the **connected double cover** of $Sp(2n,\mathbb{R})$
- $K := \widetilde{U}(n)$ the **maximal compact** subgroup of G
  $= \{[g,z] \in U(n) \times U(1) \colon \det(g) = z^2\}$
- $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{p}_0$
- $\mathfrak{a}_0 :=$ maximal abelian subspace of $\mathfrak{p}_0$
- $M := Z_K(\mathfrak{a}_0)$
- $\Delta(\mathfrak{g}_0, \mathfrak{a}_0) = \{\pm\epsilon_k \pm \epsilon_l\}_{k,l=1\ldots n} \cup \{\pm 2\epsilon_k\}_{k=1\ldots n}$ *type C_n*
- $W \simeq S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ *all permutations and sign changes*

The group M and its genuine representations

$M = Z_K(\mathfrak{a}_0)$ subgroup of K generated by the elements

$$\boldsymbol{m_k = \left[\mathbf{diag}(1,\ldots,1,\underset{k}{-1},1,\ldots,1),\ i\right]}, \; k = 1\ldots n \quad \textit{(of order 4)}$$

Genuine M-types Irreducible repr.s δ of M s.t. $\delta([I,-1]) \neq +1$.

$$\Updownarrow$$

Subsets $S \subset \{1\ldots n\}$ $m_k^2 = [I,-1] \rightarrow$ each generator m_k acts by $\pm i$

S keeps track of which generators act by $-i$

$$\boldsymbol{\delta_S(m_k) = \begin{cases} -i & \textbf{if } k \in S \\ +i & \textbf{otherwise} \end{cases}}$$

$Mp(6)$	m_1	m_2	m_3
$\delta_{\{2,3\}}$	$+i$	$-i$	$-i$

An action of the Weyl group on genuine M-types

W acts on $\widehat{M}$ ← $(s_\alpha \cdot \delta)(m) := \delta(\sigma_\alpha^{-1} m \sigma_\alpha) \quad \forall\, m \in M, \forall\, \alpha \in \Delta$

The **stabilizer** of δ in W is $\boxed{W^\delta := \{w \in W : w \cdot \delta \simeq \delta\}}$.

For all $S \subset \{1, \ldots, n\}$, set $q = |S|$, $p = |S^c|$.

- $\boldsymbol{W^{\delta_S} \simeq W(C_p) \times W(C_q)}$ ← $s_{2\epsilon_k}$ & $s_{\epsilon_k \pm \epsilon_l}$, k, l in S or S^C
- $\boldsymbol{W \cdot \delta_S = \{\delta_T : |T| = q,\ |T^c| = p\}}$

$\boxed{W\text{-orbits of genuine } M\text{-types} \leftrightsquigarrow \text{pairs } (p,q) : p, q \in \mathbb{N}, p + q = n}$

Pick representatives $\delta^{p,q} := \delta_{\{p+1,\ldots,n\}}$. $\delta^{p,q}(m_k) = \begin{cases} +i & \text{if } k \le p \\ -i & \text{if } k > p. \end{cases}$

The group K and its genuine representations

Maximal compact subgroup of G: $K = \widetilde{U}(n)$

Genuine K-types parameterized by highest weight $(a_1, \dots, a_n)$ with $a_1 \geq a_2 \geq \cdots \geq a_n$ and $a_j \in \mathbb{Z} + \frac{1}{2}, \forall j$

fine K-types	highest weight	restriction to M
$\Lambda^p(\mathbb{C}^n) \otimes \det^{-1/2}$	$(\underbrace{\tfrac{1}{2}, \dots, \tfrac{1}{2}}_{p}, \underbrace{-\tfrac{1}{2}, \dots, -\tfrac{1}{2}}_{q})$	$W \cdot \delta^{p,q}$

- If we restrict a *fine* K-type to M, we get *one full* W*-orbit* in $\widehat{M}$
- Each genuine M-type δ is contained in a *unique fine* K*-type* μ_δ.

Genuine Complementary Series of $Mp(2n)$

- $MA :=$ Levi factor of a minimal parabolic
- $\delta :=$ *genuine* irreducible representation of M
- $\nu :=$ real character of A
- $P = MAN :=$ a minimal parabolic making ν weakly dominant.

Minimal Principal Series $I_P(\delta, \nu) := \mathrm{Ind}_P^G (\delta \otimes \nu \otimes 1)$

Langlands Quotient $J(\delta, \nu)$:= composition factor of $I_P(\delta, \nu) \supseteq \mu_\delta$

δ-Complementary Series $CS(G, \delta) := \{\nu \in \mathfrak{a}_{\mathbb{R}}^* \mid J(\delta, \nu) \text{ is unitary}\}$

Problem: **Find** $\boldsymbol{CS(Mp(2n), \delta^{p,q})}$

THEOREM 1

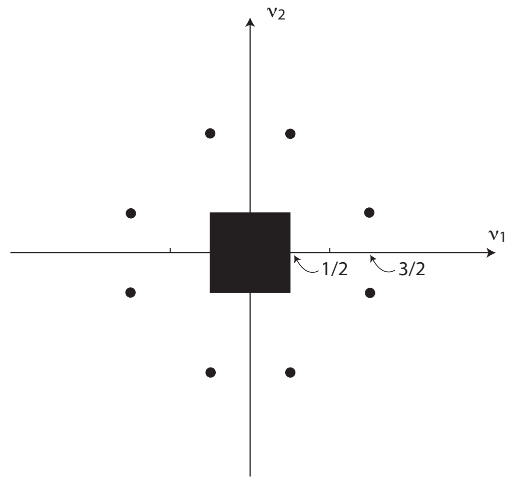
Theorem 1: For all $\nu \in \mathfrak{a}^*_{\mathbb{R}}$, write $\nu := (\nu^p | \nu^q)$. The map:

$$CS(Mp(2n), \delta^{p,q}) \to CS(SO(p+1,p)_0, 1) \times CS(SO(q+1,q)_0, 1)$$
$$\nu \mapsto (\nu^p, \nu^q)$$

is a well defined injection. (1 *denotes the trivial* M*-type*)

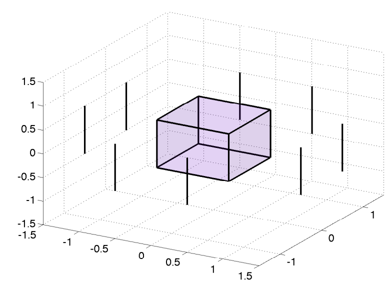
Spherical complementary series of real split orthogonal groups are known (Barbasch). Hence this theorem provides explicit necessary conditions for the unitarity of genuine principal series of $Mp(2n)$.

Example: $CS(Mp(6), \delta^{2,1}) \to CS(SO(3,2)_0, 1) \times CS(SO(2,1)_0, 1)$



$CS(SO(3,2)_0, 1)$

$CS(SO(2,1)_0, 1)$



$\Rightarrow$ $\boldsymbol{CS(Mp(6), \delta^{2,1})}$ **embeds into:**

A reformulation of THEOREM 1

For all p, $q \in \mathbb{N}$ s.t. $p+q=n$, set:

$$\boldsymbol{G^{\delta^{p,q}} \equiv SO(p+1,p)_0 \times SO(q+1,q)_0}$$

and note that $W(G^{\delta^{p,q}}) = W^{\delta^{p,q}}$.

$\boldsymbol{G^{\delta^{p,q}}}$:= *connected real split group whose root system is* *dual to the system of good roots for* $\delta^{p,q}$.

Theorem 1: **The $\delta^{p,q}$-complementary series of $Mp(2n)$ embeds into the spherical complementary series of $G^{\delta^{p,q}}$.**

Proof: based on Barbasch's idea to use calculations on petite K-types to compare unitary parameters for different groups.

Comparing unitary parameters for $Mp(2n)$ and $G^{\delta^{p,q}}$

$J(\delta^{p,q},\nu)$ unitary for $Mp(2n)$

$\Updownarrow$

$T(\mu,\delta^{p,q},\nu)$

pos. semidefinite

$\forall \mu \in \widehat{K}$

$<-$

$|$

? $|$

? $|$

$|$

$->$

$J(1,\nu)$ unitary for $G^{\delta^{p,q}}$

$\Updownarrow$

$A(\psi,1,\nu)$

pos. semidefinite

$\forall \psi \in \widehat{W^{\delta^{p,q}}}$

$\Updownarrow$

$A(\psi,1,\nu)$

pos. semidefinite

$\forall \psi \in \widehat{W^{\delta^{p,q}}}$ *relevant*

A matching of operators

Key Proposition:

$\forall$ ***relevant*** $\boldsymbol{W^{\delta^{p,q}}}$**-type** $\boldsymbol{\psi}$, $\exists$ **a "petite"** $\boldsymbol{K}$**-type** $\boldsymbol{\mu}$ **s.t.**

$$\underbrace{\boldsymbol{T(\mu, \delta^{p,q}, \nu)}}_{operator\ for\ Mp(2n)} \quad = \quad \underbrace{\boldsymbol{A(\psi, 1, \nu)}}_{operator\ for\ G^{\delta^{p,q}}}$$

Sketch of the proof:

- $T(\mu, \delta^{p,q}, \nu)$ is defined on $\mathrm{Hom}_M(\mu, \delta^{p,q})$
- This space carries a representation ψ_μ of $W^{\delta^{p,q}} \longleftarrow = W(G^{\delta^{p,q}})$
- Attached to ψ_μ, $\exists$ a spherical operator $A(\psi_\mu, 1, \nu)$ for $G^{\delta^{p,q}}$
- If μ is petite, $T(\mu, \delta^{p,q}, \nu) = A(\psi_\mu, 1, \nu)$
- For all $\psi \in \widehat{W^{\delta^{p,q}}}$ relevant, $\exists$ $\mu \in \widehat{K}$ petite such that $\psi = \psi_\mu$.$\square$

A matching of relevant $W^{\delta^{p,q}}$-types with petite K-types

$((p-s)\times(s))\otimes triv$	$\Big(\underbrace{\tfrac{1}{2},\dots,\tfrac{1}{2}}_{p-s},\underbrace{-\tfrac{1}{2},\dots,-\tfrac{1}{2}}_{q},\underbrace{-\tfrac{3}{2},\dots,-\tfrac{3}{2}}_{s}\Big)$
$(p-s,s)\otimes triv$	$\Big(\underbrace{\tfrac{3}{2},\dots,\tfrac{3}{2}}_{s},\underbrace{\tfrac{1}{2},\dots,\tfrac{1}{2}}_{p-2s},\underbrace{-\tfrac{1}{2},\dots,-\tfrac{1}{2}}_{q+s}\Big)$
$triv\otimes((q-r)\times(r))$	$\Big(\underbrace{\tfrac{3}{2},\dots,\tfrac{3}{2}}_{r},\underbrace{\tfrac{1}{2},\dots,\tfrac{1}{2}}_{p},\underbrace{-\tfrac{1}{2},\dots,-\tfrac{1}{2}}_{q-r}\Big)$
$triv\otimes(q-r,r)$	$\Big(\underbrace{\tfrac{1}{2},\dots,\tfrac{1}{2}}_{p+r},\underbrace{-\tfrac{1}{2},\dots,-\tfrac{1}{2}}_{q-2r},\underbrace{-\tfrac{3}{2},\dots,-\tfrac{3}{2}}_{r}\Big)$

$J(\delta^{p,q},\nu)$ unitary for $Mp(2n)$ $\Updownarrow$

$T(\mu,\delta^{p,q},\nu)$ pos. semidefinite $\forall \mu \in \widehat{K}$

$\Downarrow$

$T(\mu,\delta^{p,q},\nu)$ pos. semidefinite $\forall \mu \in \widehat{K}$ *petite*

$J(1,\nu)$ unitary for $G^{\delta^{p,q}}$ $\Updownarrow$

$A(\psi,1,\nu)$ pos. semidefinite $\forall \psi \in \widehat{W^{\delta^{p,q}}}$

$\Updownarrow$

$A(\psi,1,\nu)$ pos. semidefinite $\forall \psi \in \widehat{W^{\delta^{p,q}}}$ *relevant*

==>

$\uparrow$

$|$

$\forall\, \psi \in \widehat{W^{\delta^{p,q}}}$ relevant, $\exists\ \mu \in \widehat{K}$ petite s.t. $A(\psi,1,\nu) = T(\mu,\delta^{p,q},\nu)$

Non-unitarity certificates

Let $G^{\delta^{p,q}} = SO(p+1,p)_0 \times SO(q+q,q)_0$. For all $\nu = (\nu^p|\nu^q)$:

$J(\delta^{p,q},\nu)$ unitary for $Mp(2n)$ ==> $J(1,\nu)$ unitary for $G^{\delta^{p,q}}$.

The spherical unitary dual of split orthogonal groups is known. So we get ***non-unitarity certificates*** *for genuine L.Q.s of $Mp(2n)$.*

Theorem 1': If

- the spherical L.Q. $J(1,\nu^p)$ of $SO(p+1,p)_0$ is *not unitary*, or
- the spherical L.Q. $J(1,\nu^q)$ of $SO(q+1,q)_0$ is *not unitary*

then the genuine L.Q. $J(\delta^{p,q},(\nu^p|\nu^q))$ of $Mp(2n)$ is also *not unitary*.

An example of non-unitarity certificate

Let $\nu = (\nu_1, \ldots, \nu_n)$. We may assume:

$$\nu_1 \geq \cdots \geq \nu_p \geq 0 \quad \text{and} \quad \nu_{p+1} \geq \cdots \geq \nu_n \geq 0,$$

by $W^{\delta^{p,q}}$-invariance. (Recall $W^{\delta^{p,q}} = W(C_p) \times W(C_q)$.)

If *any* of the following conditions holds:

- $\boldsymbol{\nu_p > 1/2}$
- $\boldsymbol{\nu_n > 1/2}$
- $\boldsymbol{\nu_a - \nu_{a+1} > 1}$, for some a with $1 \leq a \leq p-1$, or
- $\boldsymbol{\nu_a - \nu_{a+1} > 1}$, for some a with $p+1 \leq a \leq n-1$

then the genuine Langlands quotient $J(\delta^{p,q}, \nu)$ of $Mp(2n)$ is not unitary.

An application

This non-unitarity certificate is a key ingredient in the classification of the ω-*regular* unitary dual of $Mp(2n)$.

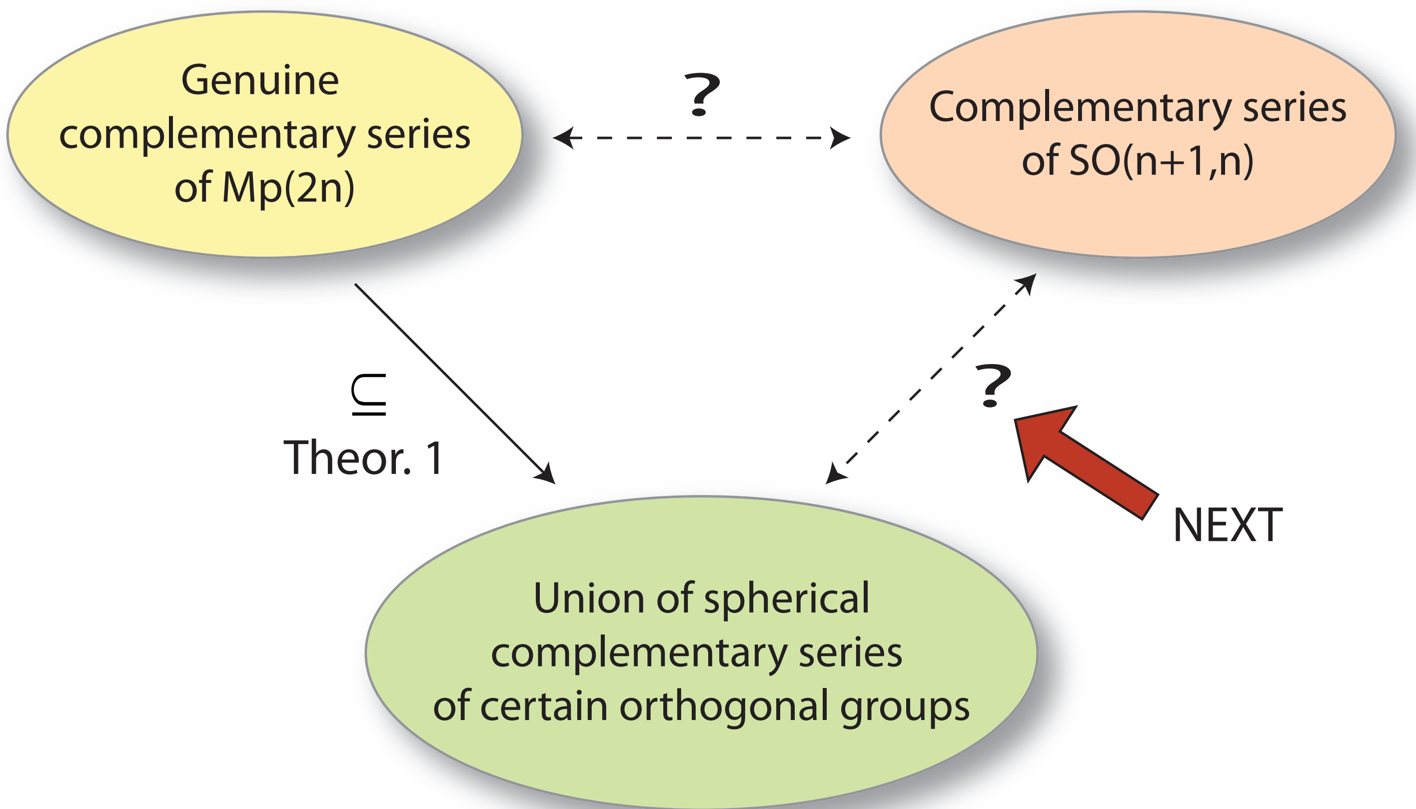
Definition: A representation of $Mp(2n)$ is called ***ω-regular*** if its infinitesimal character is at least as regular as the one of the oscillator representation.

Corollary: The only ω-*regular* complementary series repr.s of $Mp(2n)$ are the two even oscillator representations:

$$J\left(\delta_{0,n}, \left(n-\tfrac{1}{2}, \ldots, \tfrac{3}{2}, \tfrac{1}{2}\right)\right) \text{ and } J\left(\delta_{n,0}, \left(n-\tfrac{1}{2}, \ldots, \tfrac{3}{2}, \tfrac{1}{2}\right)\right).$$

PART 2

Complementary Series of $SO(n+1,n)$



NOTATION

- $G := SO(n+1,n)$
- $K := S(O(n+1)\times O(n))$ *maximal compact*
- $\Delta(\mathfrak{g}_0,\mathfrak{a}_0) = \{\pm\epsilon_k \pm \epsilon_l\} \cup \{\pm\epsilon_k\}$ *type B_n* ← dual to previous case
- $W \simeq S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$ ← same Weyl group as before
- $M := Z_K(\mathfrak{a}_0) = \{\mathrm{diag}(1, t_n, \ldots, t_1, t_1, \ldots, t_n) \colon t_j = \pm 1, \forall\, j\}$

M-types

M is generated by the elements

$$m_k = \mathrm{diag}(1,\dots,1,\underset{n+2-k}{-1},1,\dots,1,\underset{n+1+k}{-1},1,\dots,1)$$

$k = 1\dots n$ *(of order 2).*

M-types $\Leftrightarrow$ Subsets $S \subset \{1\dots n\}$ $\leftarrow$ same parametrization as before

The set S keeps track of which generators act by -1:

$$\delta_S(m_k) = \begin{cases} -1 & \text{if } k \in S \\ +1 & \text{otherwise} \end{cases}$$

$SO(4,3)$	m_1	m_2	m_3
$\delta_{\{2,3\}}$	$+1$	-1	-1

W-orbits of M-types

Just like before, we look at the action of W on $\widehat{M}$. Then

- $W^{\delta_S} \simeq W(B_p) \times W(B_q)$, for $q = |S|$, $p = |S^c|$ $\longleftarrow$ same as before
- $W \cdot \delta_S = \{\delta_T \colon |T| = q,\ |T^c| = p\}$
- W-orbits of M-types $\leftrightsquigarrow$ pairs $(p, q) \colon p, q \in \mathbb{N}, p + q = n$

$\uparrow$

same parametrization as before

Pick representatives $\delta^{p,q} := \delta_{\{p+1,\dots,n\}}$. $\delta^{p,q}(m_k) = \begin{cases} +1 & \text{if } k \leq p \\ -1 & \text{if } k > p. \end{cases}$

K-types (n even)

$K = S(O(n+1) \times O(n))$, n even

K-types $(a_1, \dots, a_{\frac{n}{2}}; b_1, \dots, b_{\frac{n}{2}})$ with $a_j,\, b_j \in \mathbb{Z},\ \forall\, j$ and

$a_1 \geq \cdots \geq a_{\frac{n}{2}} \geq 0;\ b_1 \geq \cdots \geq b_{\frac{n}{2}} \geq 0.$

If $b_{\frac{n}{2}} = 0$, there is also a sign $\epsilon = \pm 1$.

	Fine K-types	realization	res. to M
$q < \frac{n}{2}$	$(0, \dots, 0; \underbrace{1, \dots, 1}_{q}, 0, \dots, 0; +)$	$triv \otimes \Lambda^q \mathbb{C}^n$	$W \cdot \delta^{p,q}$
$q = \frac{n}{2}$	$(0, \dots, 0; 1, \dots, 1)$	$triv \otimes \Lambda^{\frac{n}{2}} \mathbb{C}^n$	$W \cdot \delta^{p,q}$
$q > \frac{n}{2}$	$(0, \dots, 0; \underbrace{1, \dots, 1}_{n-q}, 0, \dots, 0; -)$	$triv \otimes \Lambda^q \mathbb{C}^n$	$W \cdot \delta^{p,q}$

K-types (n *odd*)

$K = S(O(n+1) \times O(n))$, n *odd*

K-types

$(a_1, \dots, a_{\frac{n+1}{2}}; b_1, \dots, b_{\frac{n-1}{2}})$ with $a_j,\ b_j \in \mathbb{Z}$, $\forall\, j$ and

$a_1 \geq \cdots \geq a_{\frac{n+1}{2}} \geq 0;\ b_1 \geq \cdots \geq b_{\frac{n-1}{2}} \geq 0.$

If $a_{\frac{n+1}{2}} = 0$, *there is also a sign* $\epsilon = \pm 1$.

	Fine K-types	realization	res. to M
$q < \frac{n}{2}$	$(0,\dots,0;\underbrace{1,\dots,1}_{q},0,\dots,0;+)$	$triv \otimes \Lambda^q \mathbb{C}^n$	$W \cdot \delta^{p,q}$
$q > \frac{n}{2}$	$(0,\dots,0;\underbrace{1,\dots,1}_{n-q},0,\dots,0;-)$	$triv \otimes \Lambda^q \mathbb{C}^n$	$W \cdot \delta^{p,q}$

Complementary Series of $SO(n+1,n)$

- MA: Levi factor of a minimal parabolic
- $\delta \in \widehat{M}$
- $\nu \in \mathfrak{a}_{\mathbb{R}}^*$
- $P = MAN$:= a minimal parabolic making ν weakly dominant.

Minimal Principal Series $\boxed{I_P(\delta,\nu)}$

Langlands Quotient $\boxed{J(\delta,\nu)}$

δ-Complementary Series $\boxed{CS(SO(n+1,n),\delta)=\{\nu | J(\delta,\nu) \text{ unitary}\}}$

Problem: **Find $CS(SO(n+1,n),\delta^{p,q})$**

THEOREM 2

Theorem 2: For all $\nu \in \mathfrak{a}_{\mathbb{R}}^*$, write $\nu := (\nu^p|\nu^q)$. The map:

$$CS(SO(n+1,n),\delta^{p,q}) \longrightarrow CS(SO(p+1,p)_0,1)\text{x}CS(SO(q+1,q)_0,1)$$
$$\nu \longmapsto (\nu^p,\nu^q)$$

is a well defined injection. (1 *denotes the trivial* M*-type.*)

↑

same embedding as before

A reformulation of THEOREM 2

Set: $\boldsymbol{G^{\delta^{p,q}} \equiv SO(p+1,p)_0 \times SO(q+1,q)_0}$ ← same as before

and note that $W(G^{\delta^{p,q}}) = W^{\delta^{p,q}}$.

$\boldsymbol{G^{\delta^{p,q}}}$:= *connected real split group whose root system is equal to the system of good roots for* $\delta^{p,q}$.

Theorem 2: **The $\boldsymbol{\delta^{p,q}}$-complementary series of $\boldsymbol{SO(n+1,n)}$ embeds into the spherical complementary series of $\boldsymbol{G^{\delta^{p,q}}}$.**

Proof: based on a matching of *relevant* W-types for $G^{\delta^{p,q}}$ with *petite* K-types for $SO(n+1,n)$.

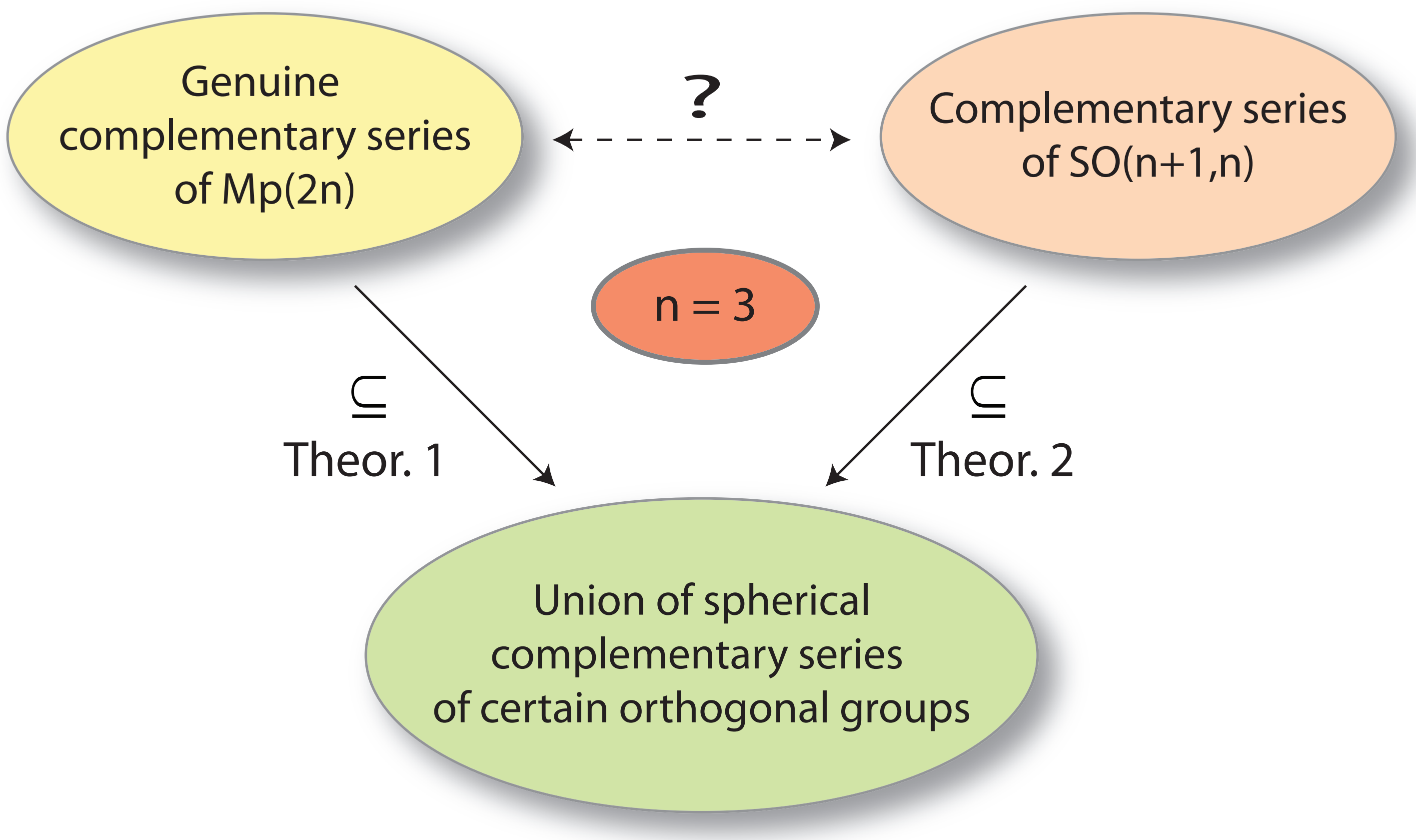
A matching of relevant $W^{\delta^{p,q}}$-types with petite K-types

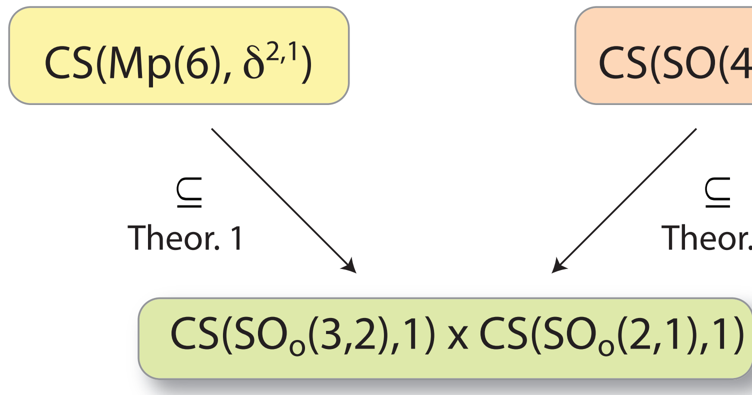
Recall that $W^{\delta^{p,q}} = W(B_p) \times W(B_q)$ and $K = S(O(n+1) \times O(n))$.

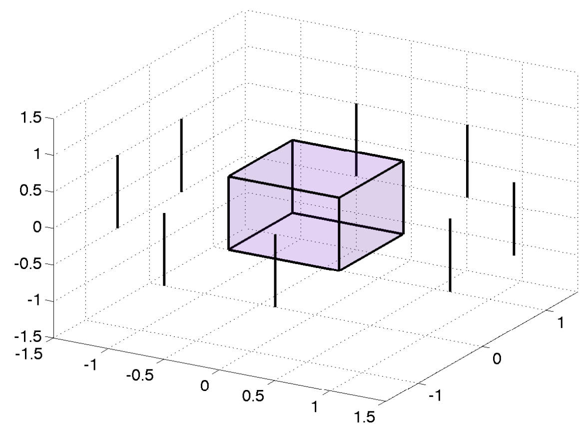
$((p-s)\times(s))\otimes triv$	$\Lambda^s(\mathbb{C}^{n+1})\otimes\Lambda^{q+s}(\mathbb{C}^n)$
$(p-s,s)\otimes triv$	an irreducible submodule of $triv\otimes[\Lambda^s(\mathbb{C}^n)\otimes\Lambda^{q+s}(\mathbb{C}^n)]$
$triv\otimes((q-r)\times(r))$	$\Lambda^r(\mathbb{C}^{n+1})\otimes\Lambda^{q-r}(\mathbb{C}^n)$
$triv\otimes(q-r,r)$	an irreducible submodule of $triv\otimes[\Lambda^r(\mathbb{C}^n)\otimes\Lambda^{q-r}(\mathbb{C}^n)]$

PART 3

An example: $n = 3$







Are these "proper containments" or "equalities"?

Are the L.Q.s $J_{Mp(6)}(\delta^{2,1}, \nu)$ and $J_{SO(4,3)}(\delta^{2,1}, \nu)$ unitary for *all* points ν of the unit cube and *all* points ν of the 8 line segments?

Unitarity of $J_{Mp(6)}(\delta^{2,1}, \nu)$ for ν in the unit cube

Theorem. The Langlands quotient $J(\delta, \nu)$ of $Mp(2n)$ is unitary for all ν in the unit cube $\{\underline{x} \in \mathfrak{a}_{\mathbb{R}}^{*} \mid 0 \leq |x_j| \leq 1/2, \forall\, j\}$.

Proof. Note that:

- For $\nu = 0$, all the operators $T(\mu, \delta, \nu)$ are positive definite.
- The signature of $T(\mu, \delta, \nu)$ can only change along the reducibility hyperplanes:

$$\begin{cases} \langle \nu, \beta \rangle \in 2\mathbb{Z} + 1 & \text{for some root } \beta \text{ which is } \textit{good} \text{ for } \delta \\ \langle \nu, \beta \rangle \in 2\mathbb{Z} \setminus \{0\} & \text{for a root } \beta \text{ which is } \textit{bad} \text{ for } \delta. \end{cases}$$

- Away from these hyperplanes, $I(\delta, \nu)$ is irreducible ($= J(\delta, \nu)$), and the operators $T(\mu, \delta, \nu)$ have constant signature. In particular, $J(\delta, \nu)$ is unitary throughout the unit cube. □

Unitarity of $J_{Mp(6)}(\delta^{2,1},\nu)$ for $\nu=\left(\frac{3}{2},\frac{1}{2}|t\right)$, $t\in[0,\frac{1}{2}]$

Theorem. The repr. $J(\delta^{p,q},\nu)$ of $Mp(2n)$ is unitary $\forall\,\nu=(\nu^p|\nu^q)$ s.t.

- $\nu^p\in CS(SO(p+1,p)_0,1)$, with $0\le|a_j|\le 3/2$ or $a_j\in\mathbb{Z}+\frac{1}{2}$
- $\nu^q\in CS(SO(q+1,q)_0,1)$, with $0\le|a_j|\le\frac{1}{2}$.

Proof. Let P_1 be a parabolic with $M_1A_1:=Mp(2p)\times\left(\widetilde{GL}(1,\mathbb{R})\right)^q$. By double induction, $J(\delta^{p,q},\nu)$ is the Langlands quotient of

$$I(\nu^q):=\operatorname{Ind}_{M_1A_1N_1}^{Mp(2n)}\left((J(\delta^{p,0},\nu^p)\otimes\delta^{0,q})\otimes\nu^q\otimes 1\right).$$

Here $J(\delta^{p,0},\nu^p)$ is a pseudopsherical repr. of $Mp(2p)$. By results of ABPTV, $J(\delta^{p,0},\nu^p)$ is unitary $\forall\,\nu^p\in CS(SO(p+1p)_0,1)$. Then the repr. $I(\nu^q)$ of $Mp(2n)$ is unitary at $\nu^q{=}0$ (unitarily induced). For all ν of interest, $I(\nu^q)$ is irreducible, hence it stays unitary by the principle of unitary deformation. $\square$

Corollary

$CS(Mp(6), \delta^{2,1})$ = $CS(SO(4,3), \delta^{2,1})$

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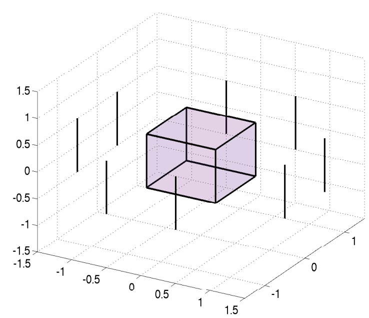
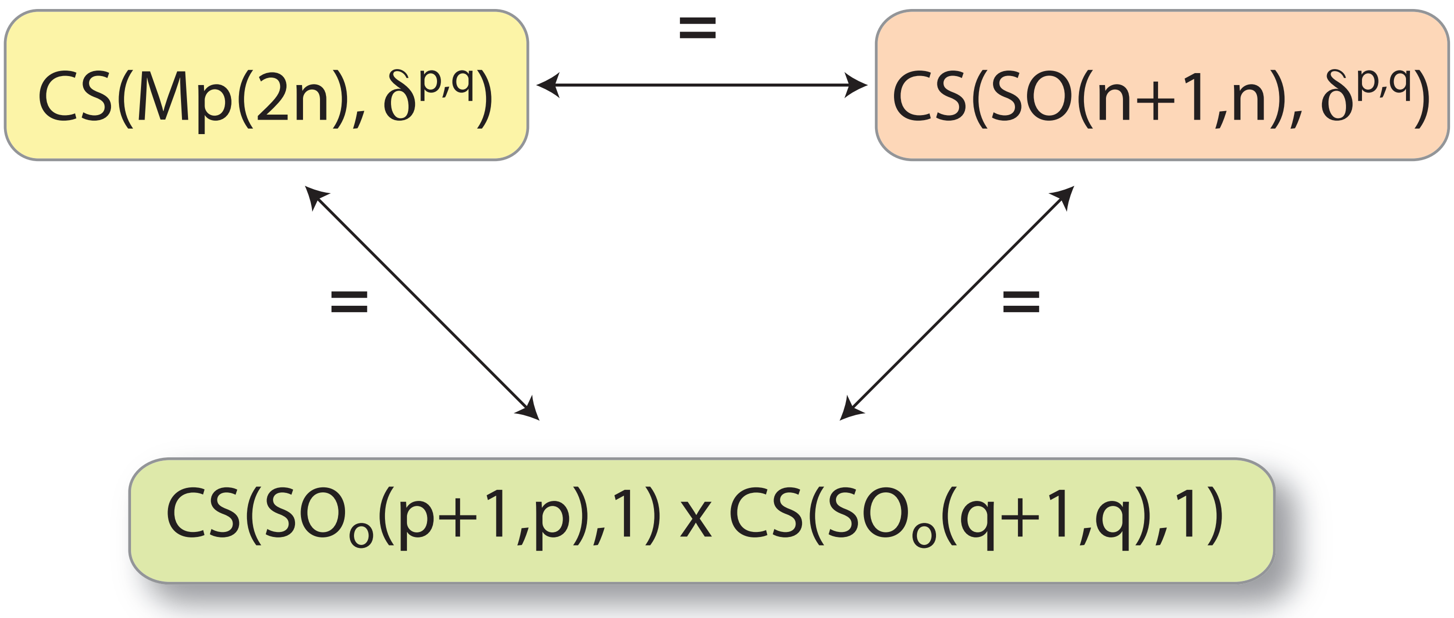


$CS(SO_o(3,2),1) \times CS(SO_o(2,1),1)$

More generally. . .

For all $\boldsymbol{n \leq 4}$ and **for all** $\boldsymbol{\delta = \delta^{p,q}}$, the following equalities hold:



PART 4

A natural conjecture

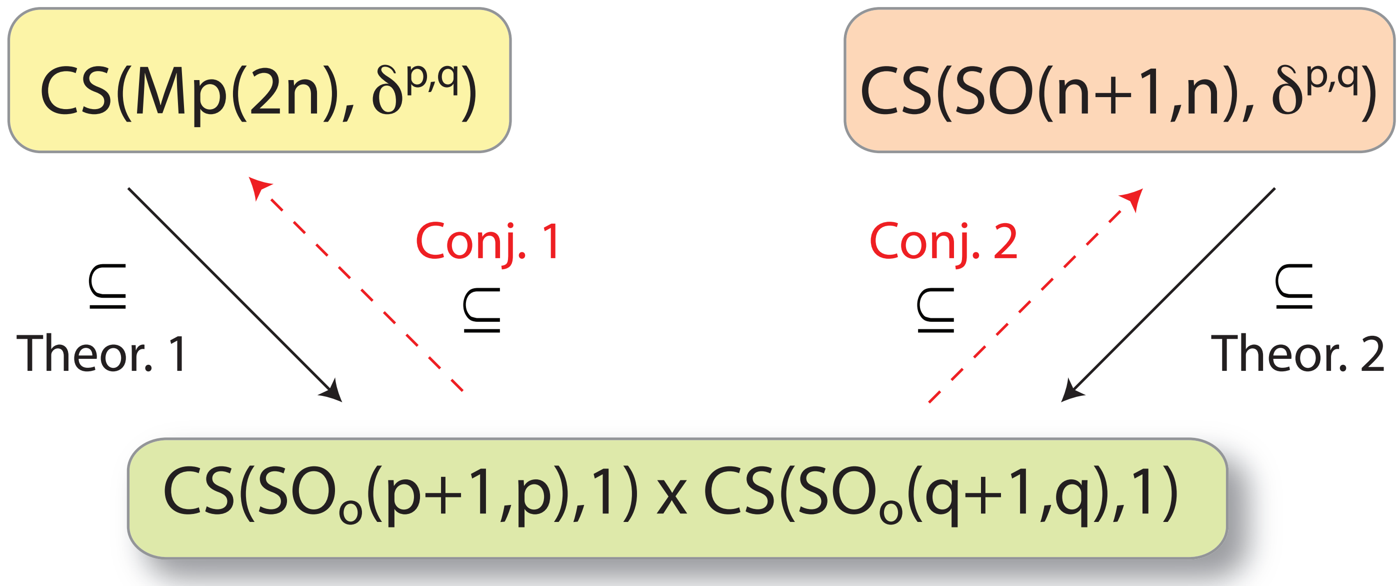
Equalities hold for all n and all choices of $\delta^{p,q}$

$$CS(Mp(2n), \delta^{p,q}) = CS(SO(n+1,n), \delta^{p,q})$$

$$CS(Mp(2n), \delta^{p,q}) = CS(SO_o(p+1,p),1) \times CS(SO_o(q+1,q),1)$$

$$CS(SO(n+1,n), \delta^{p,q}) = CS(SO_o(p+1,p),1) \times CS(SO_o(q+1,q),1)$$

Conjectures 1 and 2



Remark. We may assume $p \geq q$, because

- $J_{Mp(2n)}(\delta^{p,q}, (\nu^p|\nu^q)) = J_{Mp(2n)}(\delta^{q,p}, (\nu^p|\nu^q))^*$
- $J_{SO(n+1,n)}(\delta^{p,q}, (\nu^p|\nu^q)) = J_{SO(n+1,n)}(\delta^{q,p}, (\nu^q|\nu^p)) \otimes \chi$
  ($\chi =$ a unitary character).

(More) evidence for these conjectures

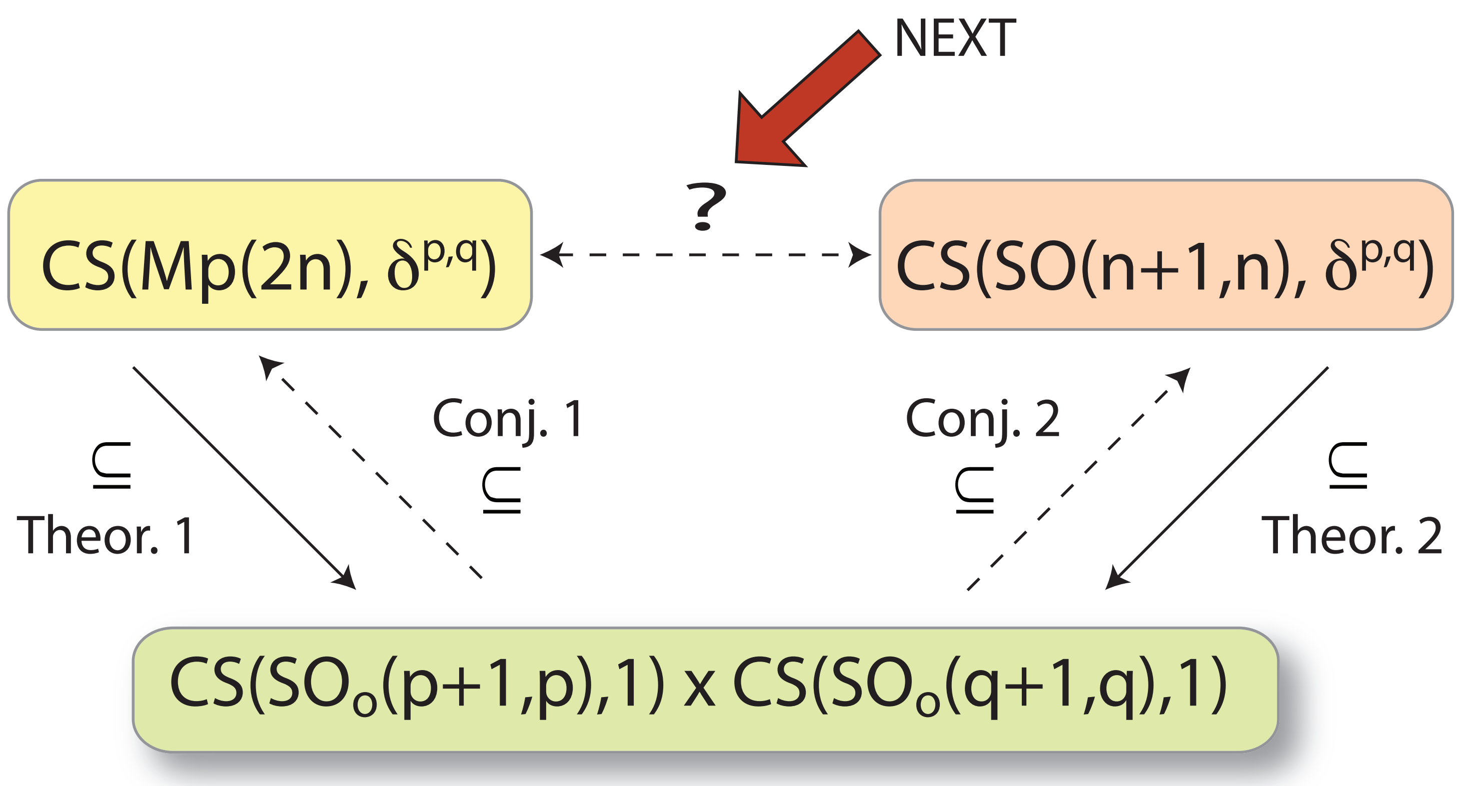
- **The case $(p, q) = (n, 0)$**

  If $(p, q) = (n, 0)$, the conjectures hold for all n. This is the pseudospherical case for $Mp(2n)$ and the spherical case for $SO(n+1, n)$. (For $Mp(2n)$, the result is due to ABPTV; for $SO(n+1, n)$, it is an empty statement.)

- **A large family of examples**

  Assume $p > q$. The conjectures hold for all $\nu = (\rho^p | \nu^q)$ with
  - ⋆ $\rho^p = \left(p - \frac{1}{2}, p - \frac{3}{2}, \ldots, \frac{3}{2}, \frac{1}{2}\right)$ = the infinitesimal character of the trivial representation of $SO(p+1, p)_0$,
  - ⋆ $\nu^q \in CS(SO(q+1, q)_0, 1)$.

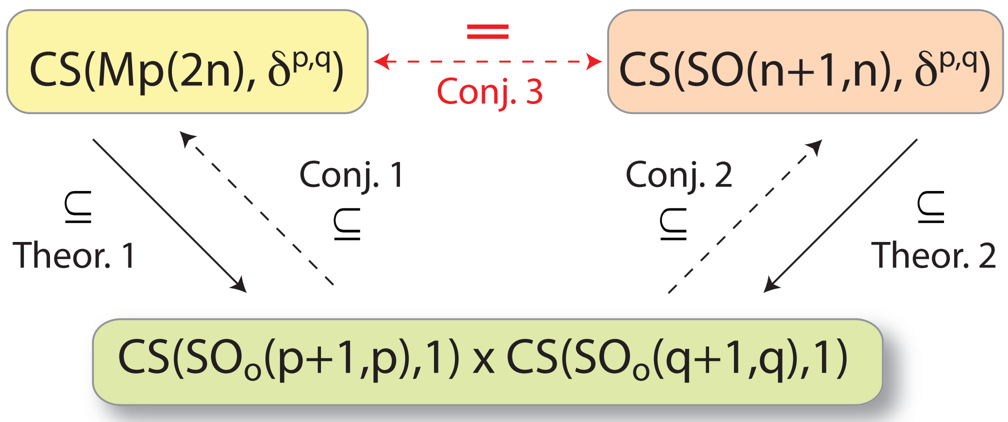
PART 5


NEXT
?
CS(Mp(2n), $\delta^{p,q}$)
CS(SO(n+1,n), $\delta^{p,q}$)
$\subseteq$ Theor. 1
Conj. 1 $\subseteq$
Conj. 2 $\subseteq$
$\subseteq$ Theor. 2
CS(SO_o(p+1,p),1) x CS(SO_o(q+1,q),1)

Conjecture 3

Conjecture 3 ***For all n and all choices of $\delta^{p,q}$:***

$$CS(Mp(2n), \delta^{p,q}) = CS(SO(n+1,n), \delta^{p,q}).$$



- Conjecture 3 is true for $n = 2$, 3, and 4.

- Conjecture 3 is independent of Conjectures 1 and 2. $\longleftarrow$ *new tools!*

θ-correspondence

Consider $G = Sp(2n, \mathbb{R})$, $G' = O(m+1, m) \subset Sp(2n(2m+1), \mathbb{R})$. Let $\tilde{G}$ and $\tilde{G}'$ be their preimages in $Mp(2n(2m+1))$:

$$\tilde{G} = Mp(2n) \qquad \tilde{G}' = \widetilde{O}(m+1, m) \text{ linear cover.}$$

- (G, G') is a *dual pair* in $Sp(2n(2m+1), \mathbb{R})$ (mutual centralizers)
- The θ-correspondence gives a bijection between certain genuine irreducible representations of $\tilde{G}$ and $\tilde{G}'$.

We can re-interpret this correspondence as a map:

$$\pi \in \widehat{Mp(2n)}_{gen} \leftrightarrow \pi' \in SO(\widehat{m+1}, m).$$

Some results of Adams, Barbasch and Li

For *all* $k \geq 0$, let $\rho_k = \left(k - \frac{1}{2}, \ldots, \frac{1}{2}\right)$. The θ-correspondence maps:

$$J_{Mp(2n)}(\delta^{p,q}, \nu) \longrightarrow J_{SO(n+k+1,n+k)}(\delta^{p+k,q}, (\rho_k|\nu))$$

$$J_{Mp(2n+2k)}(\delta^{p+k,q}, (\rho_k|\nu)) \longleftarrow J_{SO(n+1,n)}(\delta^{p,q}, \nu)$$

for all $p \geq q$.

If $k \geq n+1$, both arrows preserve unitarity. (Stable Range)

Remark: *If $k = 0$, the correspondence*

$$J_{Mp(2n)}(\delta^{p,q}, \nu) \leftrightarrow J_{SO(n+1,n)}(\delta^{p,q}, \nu)$$

is not known to preserve unitarity.

Conj.3 $J_{Mp(2n)}(\delta^{p,q}, \nu)$ unitary $\Leftrightarrow$ $J_{SO(n+1,n)}(\delta^{p,q}, \nu)$ unitary

THEOREM 3

Theorem 3: Conjecture 3 holds in each of the following cases:

(i) Conj.s $A1$ & $A2$ hold (ii) Conj.s $A1$ & $B1$ hold

(iii) Conj.s $A2$ & $B2$ hold (iv) Conj.s $B1$ & $B2$ hold.

Conjecture A

$$(\rho_{n+2}|\nu) \in CS(Mp(4n+4), \delta^{p+n+2,q})$$

Conj. A1 $\Uparrow$ $\Downarrow$ ***Conj. A2***

$$\nu \in CS(Mp(2n), \delta^{p,q})$$

Conjecture B

$$(\rho_{n+2}|\nu) \in CS(SO(2n+3, 2n+2), \delta^{p+n+2,q})$$

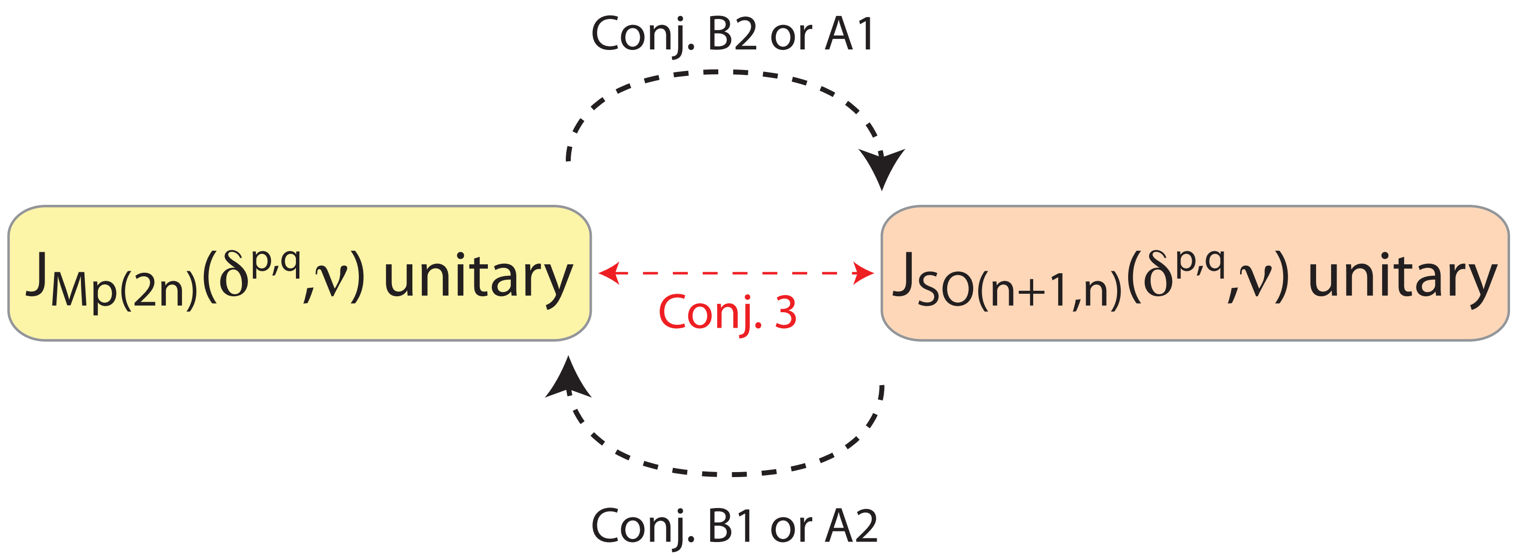
Conj. B1 $\Uparrow$ $\Downarrow$ ***Conj. B2***

$$\nu \in CS(SO(n+1, n), \delta^{p,q})$$

THEOREM 3 (a sketch of the proof)

The idea of the proof is similar to the one in ABPTV.

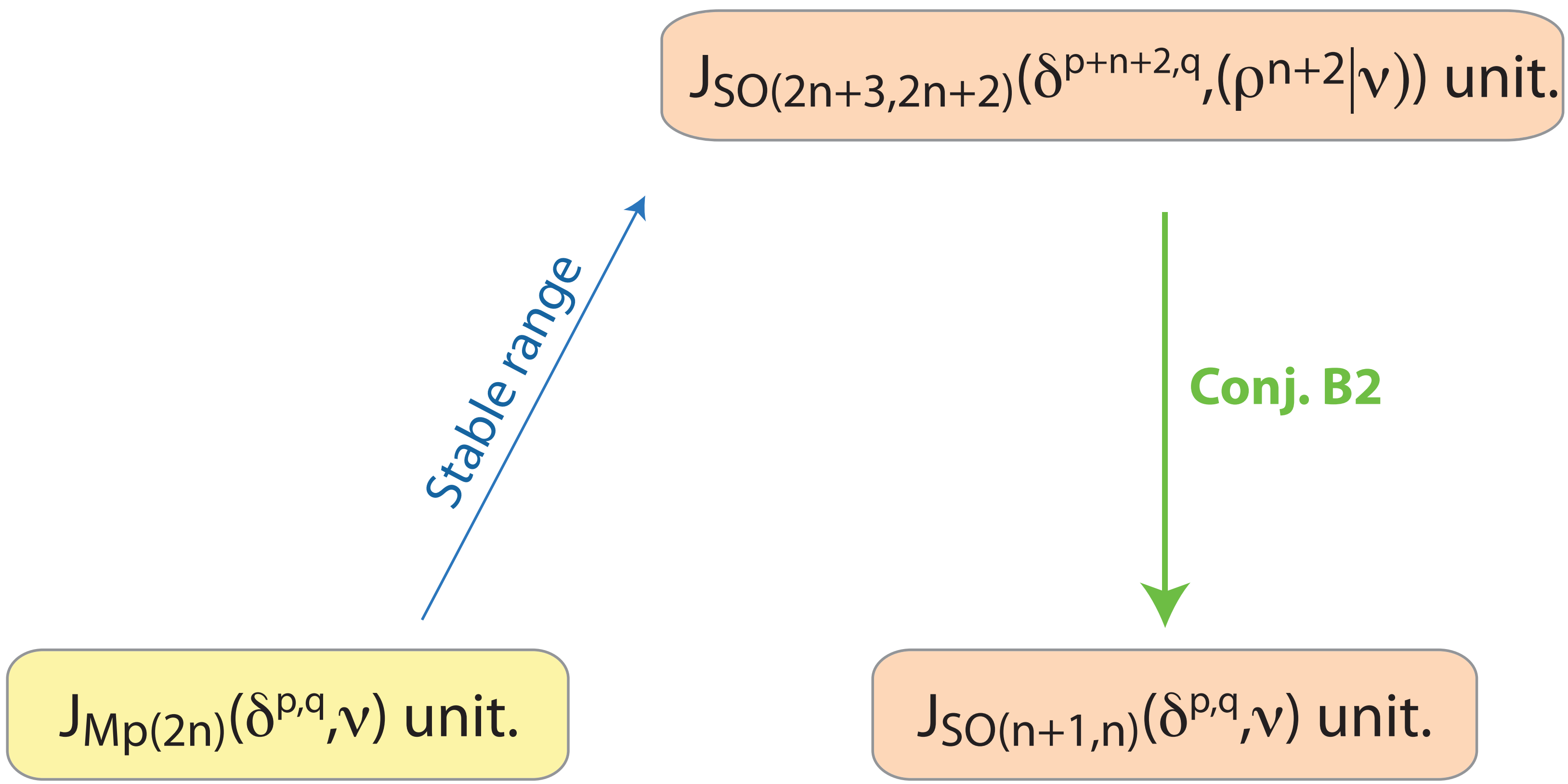
We show that:



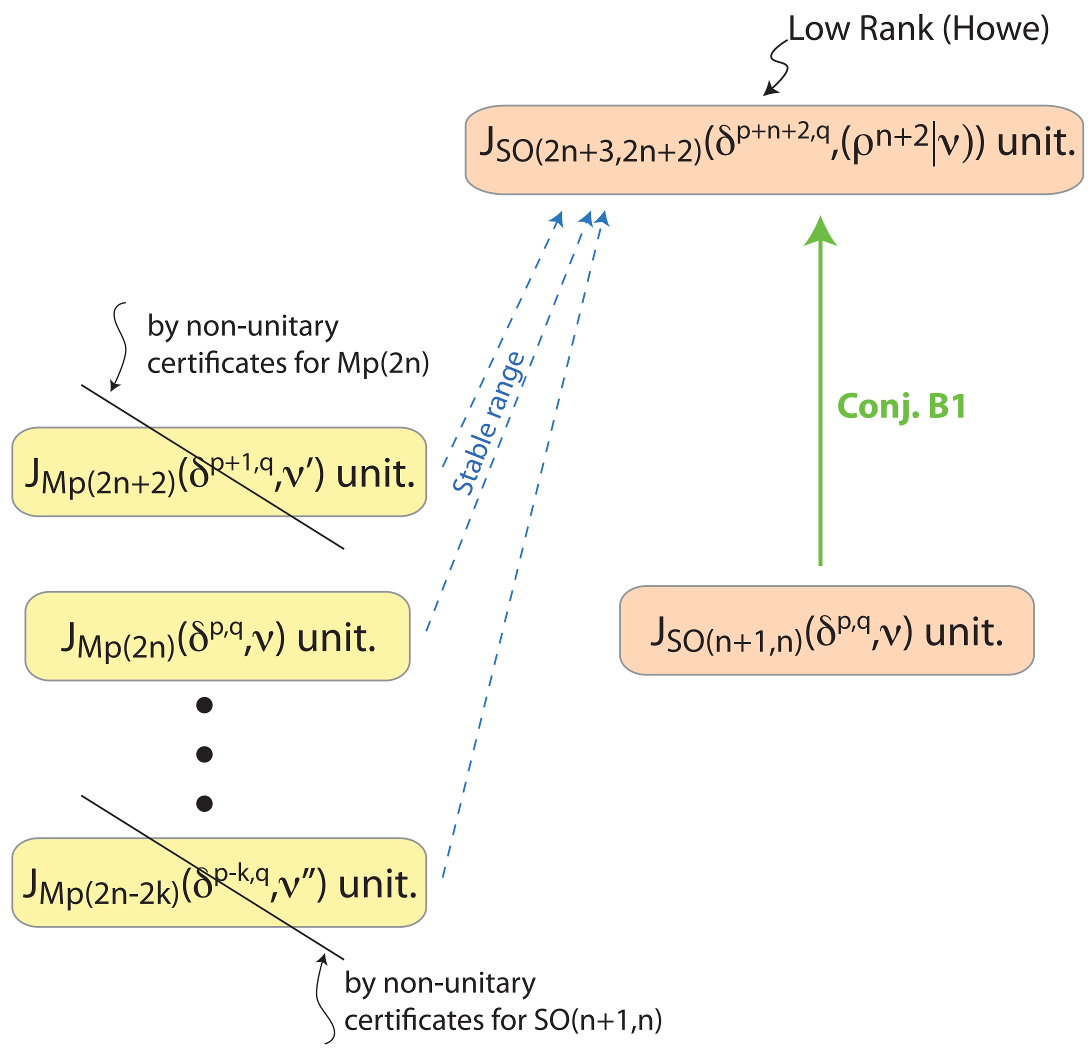
Key ingredients:

- Results on θ-correspondence (Adams, Barbasch, Li, Przebinda).
- Non-unitarity certificates for both $Mp(2n)$ and $SO(n+1,n)$.

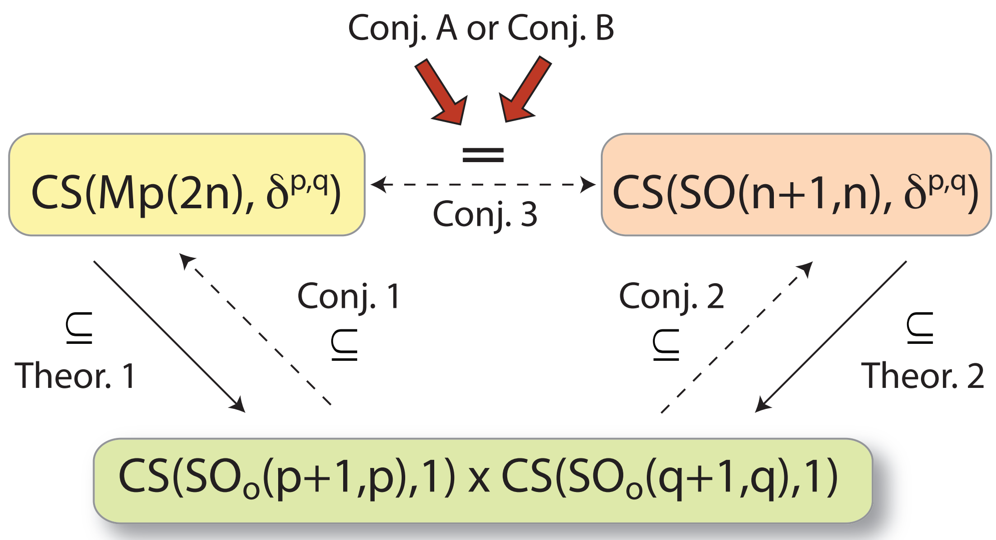
$J_{Mp(2n)}(\delta^{p,q},\nu)$ unitary $\underset{\textbf{Conj. B2}}{\Longrightarrow}$ $J_{SO(n+1,n)}(\delta^{p,q},\nu)$ unitary



$J_{Mp(2n)}(\delta^{p,q},\nu)$ unitary $\Longleftarrow$ **Conj. B1** $J_{SO(n+1,n)}(\delta^{p,q},\nu)$ unitary



Conclusions



- Conj. 1 $\Rightarrow$ Conj. A.
- Conj. 2 $\Rightarrow$ Conj. B.

If either Conj. 1 (alone) or Conj. 2 (alone) holds, then the 3 parameter sets are all equal.