PCMI LECTURE NOTES ON PROPERTY (T), EXPANDER GRAPHS AND APPROXIMATE GROUPS (PRELIMINARY VERSION)

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1. Lecture 1, Spectral gaps for infinite groups and non-amenability

The final aim of these lectures will be to prove spectral gaps for finite groups and to turn certain Cayley graphs into expander graphs. However in order to do so it is useful to have some understanding of the analogous spectral notions of amenability and Kazhdan property (T) which are important for infinite groups. In fact one important aspect of asymptotic group theory (the part of group theory concerned with studying the geometric and group theoretic properties of large finite groups) is the ability to pass from the world of infinite groups to the that of finite groups and vice-versa and to manage to transfer results from one world to the other.

We begin by reviewing the definition of amenability for a (countable) group and several of its equivalent definitions.

I. Amenability, Folner criterion.

In this lecture Γ will always denote a countable group.

Definition 1.1. *We say that* Γ *is* amenable *if there exists a sequence of finite subsets* $F_n \subset \Gamma$ *such that for every* $\gamma \in \Gamma$,

$$\frac{|\gamma F_n \Delta F_n|}{|F_n|} \to 0$$

as n *tends to infinity.*

The F_n's are called *Folner sets.* They do not need to generate Γ (in fact Γ is not assumed finitely generated). From this definition it follows easily however that Γ is the union of all $F_n F_n^{-1}$ and the $|F_n|$ tends to infinity unless Γ is finite.

The following properties can be easily deduced from this definition (exercise):

- Γ is amenable if and only if every finitely generated subgroup of Γ is amenable,
- $\mathbb{Z}^d$ is amenable
- if Γ has subexponential growth, then there is a sequence of word metric balls of radius tending to infinity which is a Folner sequence.

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We will see at the end of this section that amenability is preserved under group extensions, and thus that every solvable group is amenable.

II. Isoperimetric inequality, edge expansion

If Γ is finitely generated, say by a finite symmetric (i.e. $s \in S \Rightarrow s^{-1} \in S$) set S, then we can consider its Cayley graph $\mathcal{G}(G,S)$, which is defined to be the graph whose vertices are the elements of Γ and edges are defined by putting an edge[1] between x and y if there is $s \in S$ such that $x = ys$.

Given a finite set F of elements in Γ, we let $\partial_S F$ be the set of group elements x which are not in F but belong FS. This corresponds to the boundary of F in the Cayley graphs (i.e. points outside F but at distance at most 1 from F).

The following is straightforward (exercise):

Proposition 1.2. *The group Γ is non-amenable if and only if its Cayley graph satisfies a linear isoperimetric inequality. This means that there is $\varepsilon > 0$ such that for every finite subset F of Γ*

$$|\partial_S F| \geqslant \varepsilon |F|.$$

Exercise: Show that amenability is preserved under quasi-isometry.
Exercise: Show that the non-abelian free groups F_k are non-amenable.

It follows from the last exercise that if a countable group contains a free subgroup, it is non-amenable. The converse is not true. In fact there are finitely generated torsion groups (i.e. every element is of finite order) which are non-amenable. Adian and Novikov showed that the Burnside groups $B(n,k) := \langle a_1, \ldots, a_k | \gamma^n = 1 \ \forall \gamma \rangle$ are non-amenable for n large enough.

III. Invariant means

Amenable groups were introduced by John von Neumann in the 1930's. His definition was in terms of invariant means (amenable = admits a mean).

Definition 1.3. *An* invariant mean *on a countable group Γ is a finitely additive probability measure m defined on the set of all subsets of Γ, which is invariant under the group action by left translations, i.e. $m(\gamma A) = m(A)$ for all $\gamma \in \Gamma$ and $A \subset \Gamma$.*

It is easily checked that an invariant mean is the same thing as a continuous linear functional $m : \ell^\infty(\Gamma) \to \mathbb{R}$ such that

- $f \geqslant 0 \Rightarrow m(f) \geqslant 0$,
- $\forall \gamma \in \Gamma$, $\gamma_* m = m$,
- $m(\mathbf{1}) = 1$.

Folner showed the following:

[1]We allow multiple edges between two distinct points, but no loop at a given vertex.

Proposition 1.4. *(Folner criterion) A group Γ is amenable (in the sense of I. above) if and only if admits an invariant mean.*

The proof of the existence of the invariant mean from the Folner sequence follows by taking a weak-$\star$ limit in $\ell^\infty(\Gamma)$ of the "approximately invariant" probability measures $\frac{1}{|F_n|}\mathbf{1}_{F_n}$. For the converse, one needs to approximate m in the weak topology by functions in ℓ^1, then take appropriate level sets of these functions. For the details, we refer the reader to the appendix of the book Bekka-delaHarpe-Valette [1]. We will also give an alternative argument for the converse in the exercises using Tarski's theorem on paradoxical decompositions.

IV. Random walks on groups, the spectral radius and Kesten's criterion.

In his 1959 Cornell thesis [2], Kesten studied random walks on Cayley graphs of finitely generated groups and he established yet another characterization of amenability in terms the rate of decay of the probability of return and in terms of the spectrum of the Markov operator associated to the random walk.

Before we state Kesten's theorem, let us first give some background on random walks on groups. This will be useful later on in Lectures 3 and 4 when we discuss the Bourgain-Gamburd method.

Suppose Γ is finitely generated and μ is a finitely supported symmetric (i.e. $\forall \gamma \in \Gamma, \mu(\gamma) = \mu(\gamma)$) probability measure on Γ whose support generates Γ.

We can associate to μ an operator P_μ on $\ell^2(\Gamma)$, *the Markov operator*, by setting for $f \in \ell^2(\Gamma)$

$$P_\mu f(x) = \sum_{\gamma \in \Gamma} f(\gamma^{-1}x)\mu(\gamma).$$

Clearly P_μ is self-adjoint (because μ is assumed symmetric) and moreover

$$P_\mu \circ P_\nu = P_{\mu * \nu},$$

for any two probability measures μ and ν on Γ, where $\mu * \nu$ denotes the convolution of the two measures, that is the new probability measure defined by

$$\mu * \nu(x) := \sum_{\gamma \in \Gamma} \mu(x\gamma^{-1})\nu(\gamma).$$

The convolution is the image of the product $\mu \otimes \nu$ under the product map $\Gamma \times \Gamma \to \Gamma$, $(x, y) \mapsto xy$ and is the probability distribution of the product random variable XY, if X is a random variable taking values in Γ with distribution μ and Y is a random variable with distribution ν.

The probability measure μ induced a *random walk* on Γ, i.e. a stochastic process $(S_n)_{n \geqslant 1}$ defined as

$$S_n = X_1 \cdot \ldots \cdot X_n,$$

where the X_i's are independent random variables with the same probability distribution μ on Γ. The process $(S_n)_n$ is a Markov chain and $p_{x\to y} := \mu(x^{-1}y)$ are the transition probabilities.

When μ is the probability measure

$$\mu = \mu_S := \frac{1}{|S|}\sum_{s\in S}\delta_s,$$

where δ_s is the Dirac mass at $s \in S$ and S is a finite symmetric generating set for Γ, we say that μ and its associated process (S_n) is the *simple random walk* on (Γ, S). It corresponds to the nearest neighbor random walk on the Cayley graph $\mathcal{G}(\Gamma, S)$, where we jump at each stage from one vertex to a neighorhing vertex with equal probability.

Kesten was the first to understand that studying the probability that the random walk returns to the identity after time n could be useful to classify infinite groups. This quantity is

$$Proba(S_n = 1) = \mu^n(1),$$

where we have denoted the n-th convolution product of μ with itself by $\mu^n := \mu * \ldots * \mu$.

We will denote the identity element in Γ sometimes by 1 sometimes by e.

Proposition 1.5. *Here are some basic properties of μ^n.*

- *$\mu^{2n}(1)$ is non-decreasing,*
- *$\mu^{2n}(x) \leqslant \mu^{2n}(1)$ for all $x \in \Gamma$.*

Note that $\mu^{2n+1}(1)$ can be zero sometimes (e.g. the simple random walk on the free group), but $\mu^{2n}(1)$ is always positive.

The Markov operator P_μ is clearly a contraction in ℓ^2 (and in fact in all ℓ^p, $p \geqslant 1$), namely $||P_\mu|| \leqslant 1$. A basic tool in the theory of random walks on groups is the *spectral theorem* for self-adjoint operators applied to P_μ. This will yield Kesten's theorem and more.

Proof of Proposition 1.5. Observe that $P_{\mu^n} = P_\mu^n$ and that $\mu^n(x) = P_\mu^n\delta_e(x) = \langle P_\mu^n\delta_e, \delta_x\rangle$ (in $\ell^2(\Gamma)$ scalar product). Denoting P_μ by P for simplicity it follows that

$$\mu^{2(n+1)}(1) = \langle P^n\delta_e, P^{n+2}\delta_e\rangle \leqslant ||P^n\delta_e|| \cdot ||P^2P^n\delta_e|| \leqslant ||P^n\delta_e||^2 = \mu^{2n}(1).$$

and that

$$\mu^{2n}(x) = \langle P^{2n}\delta_2, \delta_x\rangle \leqslant ||P^n\delta_e|| \cdot ||P^n\delta_x|| = \mu^{2n}(1),$$

where the last equality follows from the fact that $P^n\delta_x(y) = P^n\delta_e(yx^{-1})$. $\square$

Proposition-Definition 1.6. *(Spectral radius of the random walk) The* spectral radius *$\rho(\mu)$ of the Markov operator P_μ action on $\ell^2(\Gamma)$ is called the spectral radius of the random walk.*

Note that since P_μ is self-adjoint, its spectral radius coincides with its operator norm $||P_\mu||$, and with $\max\{|t|, t \in spec(P_\mu)\}$.

Let us apply the spectral theorem for self adjoint operators to P_μ. This gives a resolution of identity $E(dt)$ (measure taking values into self-adjoint projections), and a probability measure $\eta(dt) : \langle E(dt)\delta_e, \delta_e\rangle$ on the interval $[-1,1]$ such that for all $n \geqslant 1$,

$$\langle P^n \delta_e, \delta_e \rangle = \int_{[-1,1]} t^n \eta(dt). \tag{1.6.1}$$

Definition 1.7. *(Spectral measure) The* spectral measure *of the random walk is the measure η associated to the Markov operator P_μ by the spectral theorem as above.*

Exercise: Show that $\langle E(dt)\delta_x, \delta_x\rangle = \eta(dt)$ for all $x \in \Gamma$ and that the other spectral measures $\langle E(dt)f, g\rangle$, with $f, g \in \ell^2(\Gamma)$ are all absolutely continuous w.r.t η.

We can now state:

Theorem 1.8. *(Kesten) Let Γ be a finitely generated group and μ a symmetric probability measure with finite support generating Γ.*

- *$\forall n \geqslant 1$, $\mu^{2n}(1) \leqslant \rho(\mu)^{2n}$ and $\lim_{n\to+\infty}(\mu^{2n}(1))^{\frac{1}{2n}} = \rho(\mu)$,*
- *(Kesten's criterion) $\rho(\mu) = 1$ if and only if Γ is amenable.*

Proof of the first item. The existence of the limit and the upper bound follows from the subadditive lemma (i.e. if a sequence $a_n \in \mathbb{R}$ satisfies $a_{n+m} \leqslant a_n + a_m$, for all $n, m \in \mathbb{N}$, then $\frac{a_n}{n}$ converges to $\inf_{n\geqslant 1} \frac{a_n}{n}$). Indeed $\mu^{2(n+m)}(1) \geqslant \mu^{2n}(1)\mu^{2m}(1)$ (the chance to come back at 1 at time $2n+2m$ is at least the chance to come back at time $2n$ and to come back again at time $2n+2m$). Take logs.

In order to identify the limit as the spectral radius, we apply the spectral theorem to P_μ, and $\mu^{2n}(1)^{\frac{1}{2n}}$ coincides with $\langle P^{2n}\delta_e, \delta_e\rangle^{\frac{1}{2n}}$, which by (1.6.1) takes the form

$$\Big(\int_{[-1,1]} t^{2n}\eta(dt)\Big)^{\frac{1}{2n}}.$$

However when $n \to +\infty$, this tends to $\max\{|t|, t \in spec(P_\mu)\} = \rho(\mu)$. □

Below we sketch a proof of Kesten's criterion via an analytic characterization of amenability in terms of Sobolev inequalities.

Proposition 1.9. *Let Γ be a group generated by a finite symmetric set S and let μ a symmetric probability measure whose support generates Γ. The following are equivalent:*

1. *Γ is non-amenable,*
2. *there is $C = C(S) > 0$ such that $||f||_2 \leqslant C||\nabla f||_2$ for every $f \in \ell^2(\Gamma)$,*
3. *there is $\varepsilon = \varepsilon(S) > 0$ such that $\max_{s\in S} ||s\cdot f - f||_2 \geqslant \varepsilon||f||_2$ for all $f \in \ell^2(\Gamma)$,*
4. *$\rho(\mu) < 1$.*

Here ∇f is the function on the set of edges of the Cayley graph of Γ associated with S given by

$$\nabla f(e) = |f(e^+) - f(e^-)|,$$

where e^+ and e^- are the extremities of the edge e.

Proof. Note that condition (3) does not depend on the generating set (only the constant ε may change). For the equivalence between (3) and (4) observe further that a finite collection of unit vectors in a Hilbert space average to a vector of norm strictly less than 1 if and only if the angle between at least two of them is bounded away from zero (and the bounds depend only on the number of vectors).

The equivalence between (2) and (3) is clear because $||\nabla f||_2$ is comparable (up to multiplicative constants depending on the size of S only) to $\max_{s\in S}||s\cdot f - f||_2$.

Condition (2) easily implies (1), because the linear isoperimetric inequality $|\partial_S F| \geqslant \varepsilon|F|$ is immediately derived from (2) by taking $f = \mathbf{1}_F$ the indicator function of F.

The only less obvious implication is (1) $\Rightarrow$ (2) as we need to go from sets to arbitrary functions. The idea to do this is to express f as a sum of indicator functions of sublevel sets. Namely, for $t \geqslant 0$, let $A_t = \{\gamma \in \Gamma; f(\gamma) > t\}$. Then for $x \in \Gamma$

$$f(x) = \int_0^{+\infty} \mathbf{1}_{t<f(x)} dt = \int_0^{+\infty} \mathbf{1}_{A_t}(x) dt$$

and, say if $f(e^+) > f(e^-)$ for an edge e of $\mathcal{G}(\Gamma, S)$

$$|f(e^+) - f(e^-)| = \int_0^{+\infty} \mathbf{1}_{f(e^-)<t<f(e^+)} dt = \int_0^{+\infty} \mathbf{1}_{\partial A_t}(e) dt$$

where ∂A (for any subset $A \subset \Gamma$) is the set of edges connecting a point in A to a point outside A.

Summing over all vertices x and all edges e we obtain the *co-area formulae*:

$$\int_0^{+\infty} |A_t| dt = ||f||_1$$

and

$$\int_0^{+\infty} |\partial_e A_t| dt = ||\nabla f||_1$$

If Γ is non-amenable, it satisfies the linear isoperimetric inequality and therefore there is $\varepsilon > 0$ such that $|\partial A| \geqslant \varepsilon|A|$ for every finite subset A of Γ. Applying this to A_t and using the co-area formulae, we conclude:

$$||\nabla f||_1 \geqslant \varepsilon ||f||_1$$

for every ℓ^1 function on Γ. To get the ℓ^2 estimate, simply note that $||f||_2^2 = ||f^2||_1$ and

$$||\nabla f^2||_1 \leqslant 2||\nabla f||_2||f||_2$$

for every $f \in \ell^2$ as one can see by applying the Cauchy-Schwartz inequality (combined with $|x^2 - y^2| = |x-y||x+y|$ and $|x+y|^2 \leqslant 2(x^2+y^2)$). □

Exercise: if $1 \to \Gamma_1 \to \Gamma_2 \to \Gamma_3 \to 1$ is an exact sequence, then Γ_2 is amenable if and only if Γ_1 and Γ_3 are amenable [hint: one can use property (3) as a working definition for amenability].

V. Linear groups and the Tits alternative

... to be continued ...

References

[1] B. Bekka, P. de la Harpe and A. Valette, *Kazhdan's property* (T) New Mathematical Monographs, 11. Cambridge University Press, Cambridge, 2008. xiv+472 pp.

[2] H. Kesten, *Symmetric random walks on groups,* Trans. Amer. Math. Soc., **92** (1959), 336–354.

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