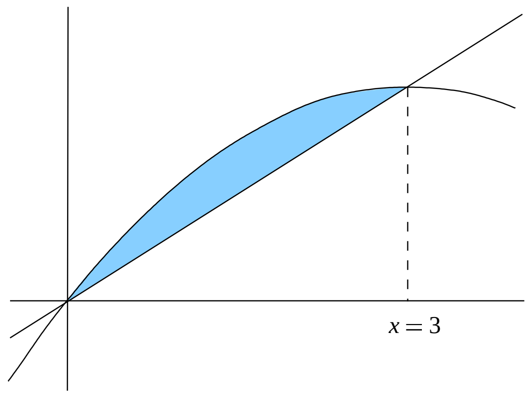
Calculus III
Practice Problems 8: Answers

1. What is the mass of the lamina bounded by the curves $y = 3x$ and $y = 6x - x^2$ where the density function is $\delta(x,y) = xy$?

Answer. Let R represent the region covered by the lamina.



From the figure we see that R is a type 1 domain, bounded above by $y = 6x - x^2$ and below by $y = 3x$ over the range $0 \le x \le 3$. (To see that, solve $6x - x^2 = 3x$ for x). Thus the mass is

$$Mass = \int\int_R \delta dA = \int_0^3 [\int_{3x}^{6x-x^2} xydy]dx\ .$$

The inner integral is

$$\int_{3x}^{6x-x^2} xydy = x\frac{y^2}{2}\Big|_{3x}^{6x-x^2} = \frac{x}{2}(36x^2 - 12x^3 + x^4 - 9x^2) = \frac{1}{2}(x^5 - 12x^4 + 27x^3)$$

Thus

$$Mass = \frac{1}{2}\int_0^3 (x^5 - 12x^4 + 27x^3)dx = \frac{1}{2}[\frac{x^6}{6} - \frac{12x^5}{5} + \frac{27x^4}{4}]_0^3 = \frac{81}{2}(\frac{9}{6} - \frac{36}{5} + \frac{27}{4})$$

which is $(81/2)(21/20) = 42.525$.

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2. A lamina filled with a homogeneous material (the density is identicaly equal to 1) is in the shape of the region $0 \le x \le \pi$, $0 \le y \le \sin x$. Find its center of mass.

Solution.

$$Mass = \int_0^\pi \int_0^{\sin x} dydx = \int_0^\pi \sin x dx = 2.$$

Now, by symmetry, the x-coordinate of the center of mass is $\bar{x} = \pi/2$. To find $\bar{y}$, we calculate

$$Mom_{y=0} = \int_0^\pi \int_0^{\sin x} ydydx == \frac{1}{2}\int_0^\pi \sin^2 x dx = \frac{\pi}{4}\ .$$

Thys $\bar{y} = \pi/8$ and the center of mass is $(\pi/4, \pi/8)$.

3. The surface H, given in cylindrical coordinates by $z = 2\theta$ is a helicoid. What is the volume of the region R bounded above by $H, 0 \leq \theta \leq 2\pi$, below by the plane $z = 0$ and lying over the disc $r \leq 1$?

Answer. Here we use polar coordinates. The volume is

$$\int\int_R zdA = \int_R 2\theta(rdrd\theta) = 2\int_0^1\int_0^{2\pi} r\theta rdrd\theta = 2\int_0^1 rdr\int_0^{2\pi}\theta d\theta = 2\pi^2 .$$

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4. A beach B is shaped in the form of a crescent. We model this on the area between the circle of radius 1, centered at the origin, and the circle of radius 3/4 centered at the point (3/4,0), where the units are in miles. Suppose that the human density σ decreases as we move from the beach according to $\sigma(x,y) = 1000(x^2+y^2)^{-2}$ people per square mile.What is the population on that beach?

Answer. We move to polar coordinates (r,θ) Then the crescent is the domain bounded by the circles $r = 1$ and $r = (3/2)\cos\theta$. These curves intersect when $\cos\theta = 2/3$, let $\pm\alpha$ represent those angles. Then B is given by the relations $-\alpha \leq \theta \leq \alpha,\ 1 \leq r \leq (3/2)\cos\theta$, and the population is the integral of the population density on this beach:

$$Population\ = \int\int \sigma dA = \int_{-\alpha}^{\alpha}\int_1^{\frac{3}{2}\cos\theta}\frac{10^3}{r^4}rdrd\theta .$$

The inner integral is

$$\int_1^{\frac{3}{2}\cos\theta}\frac{10^3}{r^3}dr = 10^3(-\frac{1}{2}r^{-2})|_1^{\frac{3}{2}\cos\theta} = \frac{10^3}{2}(1-\frac{4}{9}\sec^2\theta) .$$

Then

$$Population\ = \frac{10^3}{2}\int_{-\alpha}^{\alpha}(1-\frac{4}{9}\sec^2\theta)d\theta = \frac{10^3}{2}(\theta-\frac{4}{9}\tan\theta|_{-\alpha}^{\alpha} = 10^3\frac{4}{9}\tan\alpha .$$

Since $\cos\alpha = 2/3,\ \tan\alpha = 2/\sqrt{5}$, and the population is 397.5 people.

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5. The curve $z = (x-1)^2, 0 \leq z \leq 1$ is rotated about the z-axis, enclosing, together with the xy-plane, a 3-dimensional region R. R is filled with a substance whose density is inversely proportional to the distance from the z-axis. Find the total mass of this object.

Answer. The region is that under the curve (using polar coordinates) $z = (r-1)^2$ above the disc $R = \{0 \leq r \leq 1\}$ on the xy-plane. The density is $\delta = k/r$. Thus, the mass is

$$\int\int_R z\delta drd\theta = \int_0^{2\pi}[\int_0^1(r-1)^2\frac{k}{r}rdr]d\theta = 2\pi k\frac{(r-1)^3}{3}|_0^1 = \frac{2\pi k}{3}$$

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6. As (u,v) runs through the region $u^2+v^2 \leq 1$, the vector function

$$\mathbf{X}(u,v) = (u^2+v^2)\mathbf{I} + (u^2-v^2)\mathbf{J} + uv\mathbf{K}$$

describes a surface S in three space. Write down the double integral which must be calculated to find the surface area of S.

Answer. We have to integrate $dS = |\mathbf{X}_u \times \mathbf{X}_v|dudv$:

$$\mathbf{X}_u = 2u\mathbf{I} + 2u\mathbf{J} + v\mathbf{K}, \qquad \mathbf{X}_v = 2v\mathbf{I} - 2v\mathbf{J} + u\mathbf{K} ,$$

$$\mathbf{X}_u \times \mathbf{X}_v = (2v^2 + 2u^2)\mathbf{I} + (2v^2 - 2u^2)\mathbf{J} - 4uv\mathbf{K} ,$$

and the ensuing calculation leads to

$$|\mathbf{X}_u \times \mathbf{X}_v| = 2\sqrt{2}|u^2 - v^2| .$$

Thus

$$S = 2\sqrt{2}\int\int_R |u^2 - v^2| dudv ,$$

where R is the unit disc. We now switch to polar coordinates: $u^2 - v^2 = r^2(\cos^2\theta - \sin^2\theta) = r^2\cos 2\theta$, so

$$S = 2\sqrt{2}\int_0^{2\pi}\int_0^1 |\cos 2\theta| r^3 drd\theta = \frac{\sqrt{2}}{2}\int_0^{2\pi} |\cos 2\theta| d\theta = 4\sqrt{2}\int_0^{\pi/4}\cos 2\theta d\theta .$$

The last equality comes from the observation that the integral of $|\cos 2\theta|$ around the full circle is 8 times the integral of $\cos 2\theta$ through $\pi/4$ radians. Thus

$$S = 4\sqrt{2}\cdot\frac{1}{2}\frac{\sqrt{2}}{2} = 2 .$$

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7. Find the volume of the region lying above the disc $x^2 + y^2 \le 1$ in the xy-plane, and below the surface $z = \sin(\pi\sqrt{x^2 + y^2}/2)$.

Answer. Switching to polar coordinates, this is the volume bounded above by the surface $z = \sin(\pi r/2)$ lying above the disc $r \le 1$. Thus

$$V = \int\int_R zdA = \int_0^{2\pi}\int_0^1 \sin(\pi r/2) rdrd\theta = 2\pi\int_0^1 r\sin(\pi r/2)dr .$$

This integral we calculate by parts;

$$\int_0^1 r\sin(\pi r/2)dr = -\frac{2}{\pi}r\cos(\pi r/2)\Big|_0^1 + \frac{2}{\pi}\int_0^1 \cos(\pi r/2)dr$$
$$= \frac{2}{\pi}(1 + \frac{2}{\pi}\sin(\pi r/2)\Big|_0^1 = \frac{2}{\pi}(1 + \frac{2}{\pi}) .$$

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8. Find the mass of the lamina of the region R lying between the ellipses $x^2 + 4y^2 = 1$ and $x^2 + 4y^2 = 4$, where the density function is $\delta(x,y) = x^2 + y^2$.

Answer. Make the change of coordinates $u = x,\ v = 2y$. Then R corresponds to the region S in uv-space bounded by the circles of radius 1 and 2. Writing x and y in terms of u and v by $x = u, y = v/2$, we calculate the Jacobian:

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix} = \frac{1}{2} .$$

Thus

$$Mass = \int\int_R \delta dA = \int\int_R (x^2 + y^2)dxdy = \int\int_S (x^2 + y^2)\frac{\partial(x,y)}{\partial(u,v)}dudv$$
$$= \frac{1}{2}\int\int_S (u^2 + \frac{v^2}{4})dudv .$$

Now, we switch to polar coordinates in u,v space, obtaining

$$Mass = \frac{1}{2}\int_0^{2\pi}\int_1^2 (r^2\cos^2\theta + \frac{1}{4}r^2\sin^2\theta)rdrd\theta .$$

Integrating first with respect to θ, since

$$\int_0^{2\pi} \cos^2\theta d\theta = \int_0^{2\pi} \sin^2\theta d\theta = \pi \ ,$$

we finally obtain

$$Mass = \frac{5\pi}{8}\int_1^2 r^3 dr = \frac{5\pi}{8}\frac{16-1}{4} = \frac{75\pi}{32} \ .$$

We note that we could have made just one coordinate change, directly from x, y to r, θ:

$$x = r\cos\theta \quad y = \frac{r\sin\theta}{2} \ ,$$

but doing it in two steps is conceptually clearer and computationally easier.

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9. Find the area of the region R in the first quadrant bounded by the curves $y^2 = 2x$, $y^2 = 5x$, $x^2 = 4y$, $x^2 = 10y$.

Answer. Make the change of coordinates

$$u = \frac{y^2}{x} \ , \quad v = \frac{x^2}{y}$$

so that R corresponds to the region S in uv-space given by the inequalities $2 \le u \le 5, \ 4 \le v \le 10$. Then

$$Area(R) = \int\int_R dxdy = \int\int_S \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv \ .$$

To calculate the Jacobian, we have to solve for x, y in terms of u, v. After a little bit of algebra, we find

$$x = u^{1/3}v^{2/3} \ , \quad y = u^{2/3}v^{1/3} \ .$$

Now, computing the partial derivatives, we find

$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{1}{3}u^{-2/3}v^{2/3} & \frac{2}{3}u^{-1/3}v^{1/3} \\ \frac{2}{3}u^{1/3}v^{-1/3} & \frac{1}{3}u^{2/3}v^{-2/3} \end{pmatrix} = \frac{1}{9} - \frac{4}{9} = -\frac{1}{3} \ .$$

Finally

$$Area(R) = \int\int_S \left|\frac{\partial(x,y)}{\partial(u,v)}\right| dudv = \frac{1}{3}\int_2^5\int_4^{10} dudv = 6 \ .$$