Calculus III
Practice Problems 12: Answers

1. A fluid has density 3 and velocity field $\mathbf{V} = 4x\mathbf{I} + 3z\mathbf{J} - z\mathbf{K}$. Find the flux of the fluid out of the ball centered at the origin and of radius 4 through its boundary.

Answer. For a fluid with constant density δ, the flux through a surface is $\delta \int\int_S \mathbf{F}\cdot\mathbf{N}dS$. We'll use the divergence theorem; first calculating $\div \mathbf{V} = 4+3-1 = 6$. Thus

$$Flux = 3\int\int_S \mathbf{F}\cdot\mathbf{N}dS = 3\int\int\int_B 6dV = 4^4(6\pi) ,$$

since the volume of the ball of radius 4 is $(4/3)\pi(4)^3$.

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2. Let P be the parabolic cup $z = x^2 + y^2$ lying over the unit disc in the xy-plane. Let $\mathbf{F}(x,y,z) = y\mathbf{I} - x\mathbf{J} + \mathbf{K}$. Calculate

$$\int\int_P \text{curl } \mathbf{F}\cdot\mathbf{N}dS .$$

Answer. Let C be the boundary of P: the circle $x^2 + y^2 = 1, z = 1$. Then, by Stokes' theorem

$$\int\int_P \text{curl } \mathbf{F}\cdot\mathbf{N}dS = \int_C \mathbf{F}\cdot d\mathbf{X} = \int_C ydx - xdy + dz .$$

If we parametrize this circle by the equations $x = \cos\theta,\ y = \sin\theta,\ z = 1$, then (since $dz = 0$), this becomes

$$\int_0^{2\pi} (-\sin^2\theta - \cos^2\theta)d\theta - -2\pi .$$

Alternatively, we can use Green's theorem on the disc $D : x^2 + y^2 \leq 1, z = 1$, giving

$$\int_C ydx - xdy + dz = -\int\int_D dxdy = -2(Area(D)) = -2\pi ,$$

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3. Evaluate $\int\int_S \mathbf{F}\cdot\mathbf{N}dS$, where $\mathbf{F}(x,y,z) = x\mathbf{I} + y\mathbf{J} + z\mathbf{K}$ and S is the part of the paraboloid $z = 4 - x^2 - y^2$ which lies above the xy-plane.

Answer. To calculate the normal to the surface, we consider it parametrized as

$$\mathbf{X}(x,y) = x\mathbf{I} + y\mathbf{J} + (4 - x^2 - y^2)\mathbf{K} , \quad x^2 + y^2 \leq 4 .$$

Then

$$\mathbf{X}_x = \mathbf{I} - 2x\mathbf{K} , \quad \mathbf{X}_y = \mathbf{J} - 2y\mathbf{K} , \quad \mathbf{X}_x \times \mathbf{X}_y = 2x\mathbf{I} + 2y\mathbf{J} + \mathbf{K} .$$

We then calculate

$$\mathbf{F}\cdot\mathbf{N}dS = \mathbf{F}\cdot(\mathbf{X}_x \times \mathbf{X}_y)dxdy = (2x^2 + 2y^2 + z)dxdy = (4 + x^2 + y^2)dxdy ,$$

since $z = 4 - x^2 - y^2$ on S. Now, we integrate in polar coordinates:

$$\int\int_S \mathbf{F}\cdot\mathbf{N}dS = \int_0^{2\pi}\int_0^2 (4 + r^2)rdrd\theta = 2\pi(4r + \frac{r^3)}{3}\Big|_0^2 = \frac{64\pi}{3} .$$

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4. Evaluate $\int\int_S \sqrt{1+x^2+y^2}dS$ where S is the surface given parametrically by

$$\mathbf{X}(s,t) = s\cos t\mathbf{I} + s\sin t\mathbf{J} + t\mathbf{K}\,, \quad 0 \le s \le 5, 0 \le t \le \pi/2\,.$$

Answer. We calculate

$$\mathbf{X}_s = \cos t\mathbf{I} + \sin t\mathbf{J}\,, \quad \mathbf{X}_t = -s\sin t\mathbf{I} + s\cos t\mathbf{J} + \mathbf{K}\,, \quad \mathbf{X}_s \times \mathbf{X}_t = \sin t\mathbf{I} - \cos t\mathbf{J} + s\mathbf{K}\,.$$

Then

$$\sqrt{1+x^2+y^2}dS = \sqrt{1+s^2}|\mathbf{X}_s \times \mathbf{X}_t|dsdt = \sqrt{1+s^2}\sqrt{1+s^2}dsdt\,,$$
$$\int\int_S \sqrt{1+x^2+y^2}dS = \int_0^{\pi/2}\int_0^5 (1+s^2)dsdt = \frac{70\pi}{3}\,.$$

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5. Let S be the part of the plane $2x+y+3z=12$ which lies in the first octant, oriented upward. Let the boundary ∂S of S be oriented so that S is to its left. Given the vector field $\mathbf{F} = 3x\mathbf{I} + \mathbf{J} + y\mathbf{K}$, find $\int_{\partial S}\mathbf{F}\cdot d\mathbf{X}$.

Answer. The boundary consists of three lines, each of which will have to have its own parametrization. So, it may be easier to use Stokes' theorem:

$$\int_{\partial S}\mathbf{F}\cdot d\mathbf{X} = \int\int_S \text{curl }\mathbf{F}\cdot\mathbf{N}dS\,.$$

First we calculate curl $\mathbf{F} = \mathbf{I}$; that's a good sign. But now we have to calculate $\mathbf{N}dS$. The vector $2\mathbf{I}+\mathbf{J}+3\mathbf{K}$ (made of the coefficients of the defining equation) is normal to the plane, so the unit normal is

$$\mathbf{N} = \frac{2\mathbf{I}+\mathbf{J}+3\mathbf{K}}{\sqrt{14}}\,.$$

Now, if we write the equation of the plane as $z = (12-2x-y)/3$, and use the formula for dS of a graph, we have

$$dS = \sqrt{1+(-\frac{2}{3})^2+(-\frac{1}{3})^2}dxdy = \frac{\sqrt{14}}{3}dxdy\,.$$

The plane lies over the right triangle T with side lengths 6, 12 . Putting this altogether,

$$\int\int_S \text{curl }\mathbf{F}\cdot\mathbf{N}dS = \int\int_T \mathbf{I}\cdot\frac{2\mathbf{I}+\mathbf{J}+3\mathbf{K}}{\sqrt{14}}\frac{\sqrt{14}}{3}dxdy = \frac{2}{3}Area(T) = 24\,.$$

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6. Let B^+ be the half-ball $B: x^2+y^2+z^2 \le 1,\ z \ge 0$. Let $\mathbf{F}(x,y,z) = x\mathbf{I} + y\mathbf{J} + \mathbf{K}$. Let H be the hemisphere bounding B^+ above: $H: x^2+y^2+z^2 = 1,\ z \ge 0$. Calculate the flux of $\mathbf{F}$ from B^+ across H.

Answer. Since $\div\mathbf{F} = 2$, we suspect that the best way to compute this is by the divergence theorem. B^+ is bounded by H above, and by the disc D: $x^2+y^2 \le 1,\ z = 0$ below. The exterior normal to D is $-\mathbf{K}$, so the flux out of B^+ through D is

$$\int\int_D \mathbf{F}\cdot -\mathbf{K}dxdy = \int\int_D dxdy = -\pi\,.$$

Now we apply the divergence theorem:

$$\int\int_H \mathbf{F}\cdot\mathbf{N}dS + \int\int_D \mathbf{F}\cdot\mathbf{N}dS = \int\int\int \div\mathbf{F}dV = 2Vol(B^+) = \frac{4\pi}{3}\,.$$

Thus the flux across H is $4\pi/3 + \pi = 7\pi/3$.

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7. Let $\mathbf{F} = x^2\mathbf{I} + y^2\mathbf{J} + z^2\mathbf{K}$. Calculate the flux of $\mathbf{F}$ out of the sphere S of radius 3.

Answer. Let B be the ball of radius 3, and use the divergence theorem. Since $\div\mathbf{F} = 2x + 2y + 2z$,

$$Flux = \int\int_S \mathbf{F}\cdot\mathbf{N}dS = \int\int\int_B (2x + 2y + 2z)dV = 0\ .$$

The calculation is straightforward, but we get the result also by noting that the region is symmetric in each plane, and the integrand is an odd function of each variable. Of course, since the normal to the sphere is $\mathbf{X}/\rho$, we calculate that $\mathbf{F}\cdot\mathbf{N} = (x^3 + y^3 + z^3)/\rho$ is also an odd function on the sphere, so the flux must be zero.

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8. Let P be the piece of the plane $2x + y + 3z = 12$ which lies in the first octant, and let $\mathbf{F} = 3x\mathbf{I} + \mathbf{J} + y\mathbf{K}$. Calculate the flux of $\mathbf{F}$ through P from below.

Answer. Parametrize P as

$$\mathbf{X} = x\mathbf{I} + y\mathbf{J} + (4 - \frac{2}{3}x - \frac{1}{3}y)\mathbf{K}\ , \quad 0 \le x \le 6,\ 0 \le y \le 12 - 2x\ .$$

Then

$$\mathbf{X}_x = \mathbf{I} - \frac{2}{3}\mathbf{K}\ , \quad \mathbf{X}_y = \mathbf{J} - \frac{1}{3}\mathbf{K}\ , \quad \mathbf{X}_x \times \mathbf{X}_y = \frac{2}{3}\mathbf{I} + \frac{1}{3}\mathbf{J} + \mathbf{K}\ ,$$

and

$$\mathbf{F}\cdot\mathbf{N}dS = \mathbf{F}\cdot(\mathbf{X}_x \times \mathbf{X}_y)dxdy = (2x + y + \frac{1}{3})dxdy\ .$$

We can now compute in the x and y coordinates:

$$Flux = \int_0^6\int_0^{12-2x}(2x + y + \frac{1}{3})dydx = \int_0^6 (24x - 4x^2 + \frac{1}{2}(12 - 2x)^2 + 4 - \frac{2}{3}x)dx = 300.$$