

2210-90 Final Exam
Fall 2013

Name KEY

Instructions. Show all work and include appropriate explanations when necessary. Correct answers unaccompanied by work may not receive full credit. Please try to do all all work in the space provided and circle your final answer.

1. (19pts) Consider the vectors $\mathbf{u} = \mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and $\mathbf{v} = 2\mathbf{i} - \mathbf{j} - 2\mathbf{k}$.

(a) (3pts) Find $\mathbf{u} - \mathbf{v}$

3 $(\hat{i} + \hat{j} + 3\hat{k}) - (2\hat{i} - \hat{j} - 2\hat{k}) = -\hat{i} + 2\hat{j} + 5\hat{k}$

(b) (2pts) Find $\|\mathbf{u}\|$

2 $\|\bar{\mathbf{u}}\| = \sqrt{1^2 + 1^2 + 3^2} = \sqrt{11}$

(c) (3pts) Find $\mathbf{u} \cdot \mathbf{v}$

3 $\bar{\mathbf{u}} \cdot \bar{\mathbf{v}} = (1)(2) + (1)(-1) + (3)(-2) = -5$

(d) (3pts) Find the angle θ between the vectors \mathbf{u} and \mathbf{v} . Write your answer as $\theta = \cos^{-1}(x)$ for some number x .

3 $\bar{\mathbf{u}} \cdot \bar{\mathbf{v}} = \|\bar{\mathbf{u}}\| \|\bar{\mathbf{v}}\| \cos \theta$ So $-5 = 3\sqrt{11} \cos \theta$
 $\|\bar{\mathbf{v}}\| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{9} = 3$
 $\theta = \cos^{-1}\left(-\frac{5}{3\sqrt{11}}\right)$

(e) (4pts) Find $\mathbf{v} \times \mathbf{u}$

4 $\bar{\mathbf{v}} \times \bar{\mathbf{u}} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & -2 \\ 1 & 1 & 3 \end{vmatrix} = \hat{i}(-3+2) + \hat{j}(-2-6) + \hat{k}(2+1)$
 $= -\hat{i} - 8\hat{j} + 3\hat{k}$

(f) (4pts) Find an equation for the the plane that contains the point $(0, 1, 5)$ and has normal vector \mathbf{u} .

4 $\bar{\mathbf{n}} = \bar{\mathbf{u}} = \langle 1, 1, 3 \rangle$
 $x + y + 3z = d$. Since $(0, 1, 5)$ is in plane
 $d = 0 + 1 + 3(5) = 16$.
 $x + y + 3z = 16$

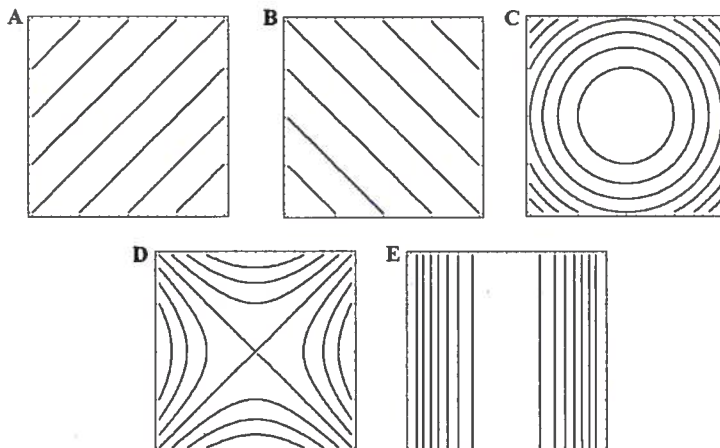
2. (6pts) Find an equation for the tangent plane to the graph of the function $f(x, y) = x^3 - y^2 + 2xy$ at the point $(1, -1, -2)$.

6 $f_x(x, y) = 3x^2 + 2y \Rightarrow f_x(1, -1) = 3 - 2 = 1$
 $f_y(x, y) = -2y + 2x \Rightarrow f_y(1, -1) = 2 + 2 = 4$
 $z = f(1, -1) + f_x(1, -1)(x-1) + f_y(1, -1)(y+1)$
 $z = -2 + 1(x-1) + 4(y+1)$

3. (10pts) Match the following functions with the plot of its level curves by writing the letter in the blank provided.

10

- C $f(x,y) = x^2 + y^2$
B $f(x,y) = x + y + 2$
D $f(x,y) = x^2 - y^2$
E $f(x,y) = x^2$
A $f(x,y) = x - y$



4. (8pts) Consider the curve

$$\mathbf{r}(t) = 2 \cos t \mathbf{i} + 5t \mathbf{j} - 2 \sin t \mathbf{k}$$

- (a) (4pts) Compute the unit tangent vector $\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$.

4

$$\mathbf{r}'(t) = -2 \sin t \hat{\mathbf{i}} + 5 \hat{\mathbf{j}} - 2 \cos t \hat{\mathbf{k}}$$

$$\|\mathbf{r}'(t)\|^2 = 4 \sin^2 t + 25 + 4 \cos^2 t = 29$$

$$\mathbf{T}(t) = \frac{1}{\sqrt{29}} (-2 \sin t \hat{\mathbf{i}} + 5 \hat{\mathbf{j}} - 2 \cos t \hat{\mathbf{k}})$$

- (b) (4pts) Compute the arc length of the curve between $t = 0$ and $t = 3$.

4

$$L = \int_0^3 \|\mathbf{r}'(t)\| dt = \int_0^3 \sqrt{29} dt = 3\sqrt{29}$$

5. (10pts) Use the method of Lagrange multipliers to find the point on the parabola $y = x^2$ in the first quadrant which is closest to the point $(0, 1)$. Do this by finding the point (x, y) in the first quadrant which minimizes the function $f(x, y) = x^2 + (y - 1)^2$, the distance squared from the point $(0, 1)$, subject to the constraint $g(x, y) = y - x^2 = 0$.

10

$$f(x, y) = x^2 + (y - 1)^2 \Rightarrow \nabla f(x, y) = \langle 2x, 2y - 2 \rangle$$

$$g(x, y) = y - x^2 \Rightarrow \nabla g(x, y) = \langle -2x, 1 \rangle$$

$$\nabla f(x, y) = \lambda \nabla g(x, y) \Rightarrow \begin{cases} 2x = -2\lambda x & \textcircled{1} \\ 2y - 2 = \lambda & \textcircled{2} \\ y - x^2 = 0 & \textcircled{3} \end{cases}$$

$$g(x, y) = 0$$

$$\textcircled{1} \Rightarrow x = 0$$

$$\Downarrow \textcircled{3}$$

$$y = 0$$

$$\Downarrow \textcircled{2}$$

$$\lambda = -2$$

$$\text{or } \lambda = -1$$

$$\text{If } \lambda = -1, \text{ then } \textcircled{2} \Rightarrow 2y - 2 = -1 \Rightarrow y = \frac{1}{2}$$

$$\text{Then } \textcircled{3} \Rightarrow x = \sqrt{\frac{1}{2}} \text{ (+ since first quadrant)}$$

So I have two points: $(0, 0)$ and $(\sqrt{\frac{1}{2}}, \frac{1}{2})$

$$\text{Now } f(0, 0) = 1$$

$$f(\sqrt{\frac{1}{2}}, \frac{1}{2}) = \frac{1}{2} + (-\frac{1}{2})^2 = \frac{3}{4}$$

So $(\sqrt{\frac{1}{2}}, \frac{1}{2})$ is closest to $(0, 1)$.

6. (24pts) Evaluate the following double or triple integrals:

(a) (6pts) $\iint_R 3x^2y \, dA$, where R is the rectangle $0 \leq x \leq 1, 0 \leq y \leq 2$.

$$= \int_0^2 \int_0^1 3x^2y \, dx \, dy = \int_0^2 (x^3y) \Big|_0^1 \, dy = \int_0^2 y \, dy = \left(\frac{y^2}{2} \right) \Big|_0^2 = \frac{4}{2} - 0 = 2$$

(b) (6pts) $\iint_D \sqrt{1+x^2+y^2} \, dA$, where D is the disk in the xy -plane of radius 2: $x^2 + y^2 \leq 4$.

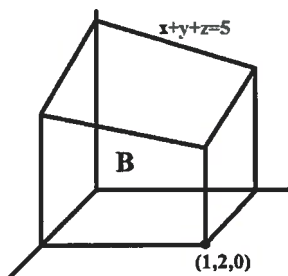
Use polar coords:

$$= \int_0^{2\pi} \int_0^2 \sqrt{1+r^2} \, r \, dr \, d\theta = 2\pi \int_0^2 \sqrt{1+r^2} \, r \, dr = \pi \int_1^5 u^{1/2} \, du$$

$$u = 1+r^2 \quad du = 2r \, dr$$

$$= \pi \left(\frac{2}{3} u^{3/2} \Big|_1^5 \right) = \frac{2\pi}{3} (5^{3/2} - 1)$$

(c) (6pts) $\iiint_B 1 \, dV$, where B is pictured solid, which is bounded by the planes $x=0, y=0, z=0, x=1, y=2$, and $x+y+z=5$. $\Rightarrow z=5-x-y$



$$= \int_0^1 \int_0^2 \int_0^{5-x-y} 1 \, dz \, dy \, dx = \int_0^1 \int_0^2 (5-x-y) \, dy \, dx$$

$$= \int_0^1 (5y - xy - \frac{y^2}{2}) \Big|_0^2 \, dx = \int_0^1 (10 - 2x - 2) \, dx$$

$$= \int_0^1 (8 - 2x) \, dx = (8x - x^2) \Big|_0^1 = 7$$

(d) (6pts) $\iiint_C (x^2 + y^2 + z^2) \, dV$, where C is the cylinder $x^2 + y^2 \leq 1, 0 \leq z \leq 2$.

Use cylindrical coords:

$$= \int_0^{2\pi} \int_0^2 \int_0^1 (r^2 + z^2) \, r \, dr \, dz \, d\theta = 2\pi \int_0^2 \int_0^1 (r^3 + z^2 r) \, dr \, dz$$

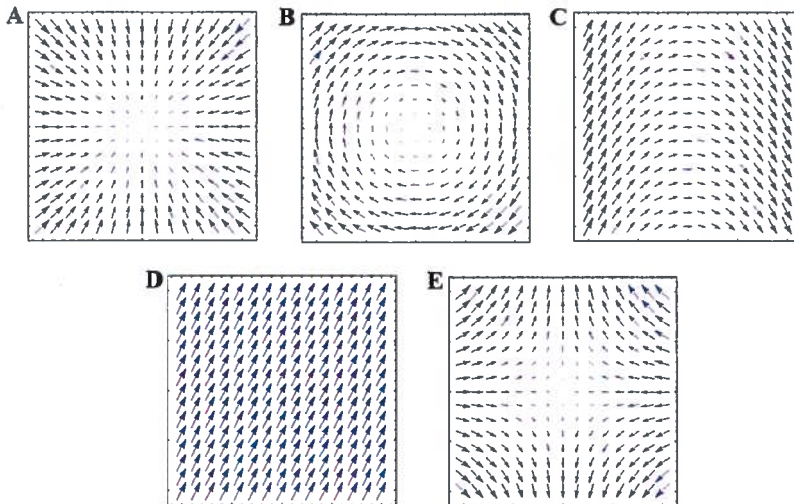
$$= 2\pi \int_0^2 \left(\frac{r^4}{4} + \frac{z^2 r^2}{2} \right) \Big|_0^1 \, dz$$

$$= 2\pi \int_0^2 \left(\frac{1}{4} + \frac{z^2}{2} \right) \, dz = 2\pi \left(\frac{z}{4} + \frac{z^3}{6} \right) \Big|_0^2$$

$$= 2\pi \left(\frac{2}{4} + \frac{8}{6} \right) = \frac{11\pi}{3}$$

7. (10pts) Match the following vector fields with the graphs below by writing the letter in the blank provided.

- 10
- B $F(x, y) = yi - xj$
 - E $F(x, y) = -xi + yj$
 - C $F(x, y) = li - xj$
 - A $F(x, y) = -xi - yj$
 - D $F(x, y) = li + 2j$



8. (16pts) Consider the vector field $F(x, y) = (2xy + 1)i + (x^2 + 2y)j$.

(a) (4pts) Compute $\text{curl } F$

4

$$\text{curl } F = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2xy+1 & x^2+2y & 0 \end{vmatrix} = \cancel{2x} (0-0) \hat{i} + (0-0) \hat{j} + (2x-2x) \hat{k} = \vec{0}$$

(b) (2pts) Is F a conservative vector field? Circle one: YES NO

(c) (6pts) Find a function $f(x, y)$ such that $\nabla f = F$.

6

$$f_x = 2xy + 1 \Rightarrow f(x, y) = \int (2xy + 1) dx = x^2 y + x + c_1(y)$$

$$f_y = x^2 + 2y \Rightarrow f(x, y) = \int (x^2 + 2y) dy = x^2 y + y^2 + c_2(x)$$

$$f(x, y) = x^2 y + x + y^2$$

(d) (4pts) Use the Fundamental Theorem of Line Integrals to compute $\int_C F \cdot dr$, where C is the curve parameterized by

$$x(t) = t \cos t \quad y(t) = \sin t + 2$$

for $0 \leq t \leq \pi/2$.

4

$$(x(0), y(0)) = (0, 2)$$

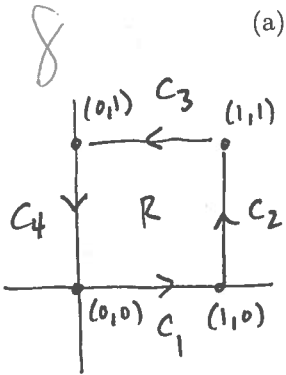
$$(x(\pi/2), y(\pi/2)) = (0, 3)$$

So

$$\int_C F \cdot dr = \int_C \nabla f \cdot dr = f(0, 3) - f(0, 2) = 9 - 4 = \boxed{5}$$

9. (16pts) Let $\mathbf{F}(x, y) = y\mathbf{i} + x^2\mathbf{j}$ and let C denote the square in the plane with vertices $(0, 0)$, $(1, 0)$, $(1, 1)$ and $(0, 1)$ traversed counterclockwise.

(a) (8pts) Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ using Green's Theorem.



$$\int_C \mathbf{F} \cdot d\mathbf{r} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = \int_0^1 \int_0^1 (2x - 1) dx dy$$

Green's Thm.

$$= \int_0^1 (x^2 - x) \Big|_0^1 dy = \int_0^1 0 dy = 0.$$

(b) (8pts) Compute the line integral $\int_C \mathbf{F} \cdot d\mathbf{r}$ directly by parameterizing the four sides of the square.

$C_1: \langle x(t), y(t) \rangle = \langle t, 0 \rangle, 0 \leq t \leq 1.$

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (0\hat{i} + t^2\hat{j}) \cdot (1\hat{i}) dt = \int_0^1 0 dt = 0.$$

$C_2: \langle x(t), y(t) \rangle = \langle 1, t \rangle, 0 \leq t \leq 1$

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (t\hat{i} + \hat{j}) \cdot (0\hat{i} + \hat{j}) dt = \int_0^1 1 dt = 1.$$

$C_3: \langle x(t), y(t) \rangle = \langle 1-t, 1 \rangle, 0 \leq t \leq 1$

$$\int_{C_3} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 (\hat{i} + (1-t)^2\hat{j}) \cdot (-1\hat{i}) dt = \int_0^1 -1 dt = -1.$$

$C_4: \langle x(t), y(t) \rangle = \langle 0, 1-t \rangle, 0 \leq t \leq 1$

$$\int_{C_4} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 ((1-t)\hat{i} + 0\hat{j}) \cdot (-\hat{j}) dt = \int_0^1 0 dt = 0.$$

So

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0 + 1 - 1 + 0 = 0.$$

10. (8pts) Consider the vector field $\mathbf{F}(x, y, z) = (x^3 + y)\mathbf{i} + (z^2 + y^3)\mathbf{j} + (z^3 + 1)\mathbf{k}$. Use the Divergence Theorem and to compute

$$\iint_S \mathbf{F} \cdot \mathbf{n} dS,$$

where S is the surface of the unit sphere $x^2 + y^2 + z^2 = 1$, and \mathbf{n} is the outward pointing unit normal vector.

$$\text{div } \mathbf{F} = \frac{\partial}{\partial x} (x^3 + y) + \frac{\partial}{\partial y} (z^2 + y^3) + \frac{\partial}{\partial z} (z^3 + 1) = 3x^2 + 3y^2 + 3z^2.$$

$$\iint_S \mathbf{F} \cdot \mathbf{n} dS \stackrel{\text{Div Thm}}{=} \iiint_B \text{div } \mathbf{F} dV = \iiint_B (3x^2 + 3y^2 + 3z^2) dV$$

Use spherical coords:

$$= \int_0^{2\pi} \int_0^\pi \int_0^1 3\rho^2 \rho^2 \sin\phi d\rho d\phi d\theta = 6\pi \left(\int_0^\pi \sin\phi d\phi \right) \left(\int_0^1 \rho^4 d\rho \right)$$

$$= 6\pi (2) (1/5) = \frac{12\pi}{5}$$

11. (8pts) Let $\mathbf{F}(x, y, z) = y \cos x \mathbf{i} + (3y + \sin z) \mathbf{j} + xyz^2 \mathbf{k}$.

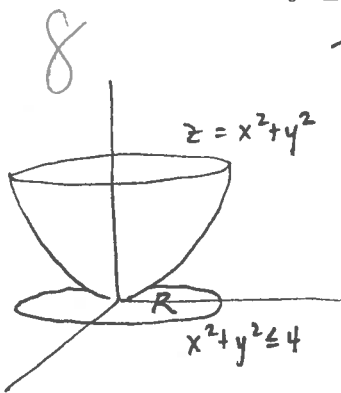
(a) (4pts) Compute $\text{div } \mathbf{F}$

4
$$\text{div } \mathbf{F} = \frac{\partial}{\partial x} (y \cos x) + \frac{\partial}{\partial y} (3y + \sin z) + \frac{\partial}{\partial z} (xyz^2) = \boxed{-y \sin x + 3 + 2xyz}$$

(b) (4pts) Compute $\text{curl } \mathbf{F}$

4
$$\text{curl } \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y \cos x & 3y + \sin z & xyz^2 \end{vmatrix} = \boxed{\hat{i} (xz^2 - \cos z) + \hat{j} (-yz^2) + \hat{k} (-\cos x)}$$

12. (8pts) Find the surface area of the piece of the paraboloid $z = x^2 + y^2$ that lies above the disk $x^2 + y^2 \leq 4$.



$f(x,y) = x^2 + y^2 \Rightarrow \nabla f(x,y) = \langle 2x, 2y \rangle$

$SA = \iint_R \sqrt{1 + f_x^2 + f_y^2} dA = \iint_R \sqrt{1 + 4x^2 + 4y^2} dA$

Use polar coordinates:

$= \int_0^{2\pi} \int_0^2 \sqrt{1 + 4r^2} r dr d\theta = 2\pi \int_0^2 \sqrt{1 + 4r^2} r dr$

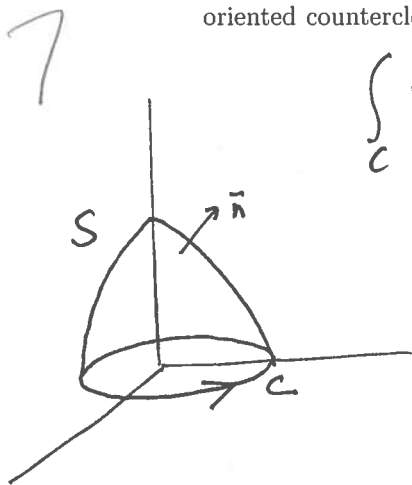
$u = 1 + 4r^2$
 $du = 8r dr$

$= \frac{\pi}{4} \int_1^{17} u^{1/2} du = \frac{\pi}{4} \left(\frac{2}{3} u^{3/2} \right) \Big|_1^{17} = \boxed{\frac{\pi}{6} (17^{3/2} - 1)}$

13. (7pts) Use Stokes's Theorem to evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$, where \mathbf{F} denotes the vector field

$\mathbf{F} = x^2 \mathbf{i} + (y + z) \mathbf{j} + (z^3 + y) \mathbf{k}$

and C is the boundary of the part of the paraboloid $z = 1 - x^2 - y^2$ which lies above the xy -plane, oriented counterclockwise when viewed from above.



$\int_C \mathbf{F} \cdot d\mathbf{r} \stackrel{\text{Stokes's Thm}}{=} \iint_S \text{curl } \mathbf{F} \cdot \mathbf{n} dS$

$$\text{curl } \mathbf{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & y + z & z^3 + y \end{vmatrix} = (1-1)\hat{i} + (0-0)\hat{j} + (0-0)\hat{k} = \mathbf{0}$$

Since $\text{curl } \mathbf{F} = \mathbf{0}$,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = 0$$