3.7 ~ Graphing polar equations

In this lesson you will:

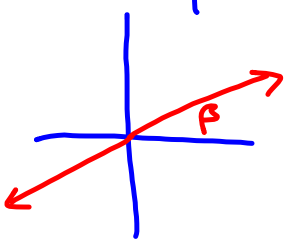
- Graph polar equations by point plotting.
- Use symmetry, zeros and maximum *r*-values to sketch graphs of polar equations.
- Recognize special polar graphs.

What do these equations represent?

These are all lines!

1. $\theta = \beta$ (β constant)

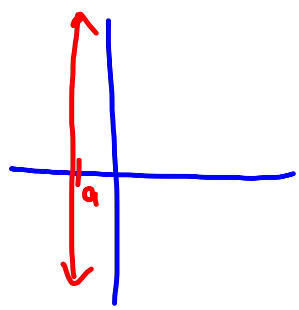
(radial line through the pole/origin)



2. $r \cos\theta = a \iff x = a$

(a constant)

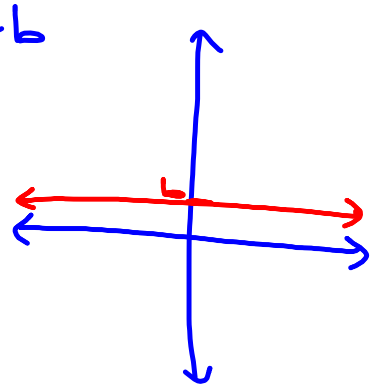
vertical line



3. $r \sin\theta = b \iff y = b$

(b constant)

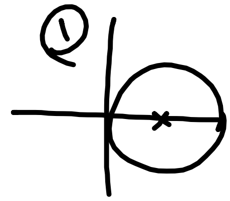
horizontal line



What about these? all of these eqns represent circles (that go thru origin)

① $r = 2a\cos\theta$ (a constant)

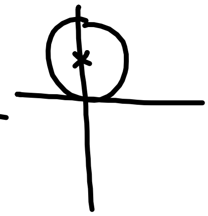
$r^2 = 2a(r\cos\theta) \iff x^2+y^2 = 2ax$

① 

$x^2 - 2ax + y^2 = 0$

② $r = 2b\sin\theta$ (b constant)

circle centered at (0,b) w/ radius b

② 

$(x^2 - 2ax + a^2) + y^2 = a^2$

$(x-a)^2 + y^2 = a^2$

③ $r = 2a\cos\theta + 2b\sin\theta$ (a,b constants)

circle centered at (a,b) w/ radius of $\sqrt{a^2+b^2}$

① circle centered at (a,0) w/ radius of a

Example 1:

$r = 4\cos\theta$

θ	0	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	π	$5\pi/4$	$3\pi/2$	$7\pi/4$	2π
r	4	$2\sqrt{2}$	2	0	-2	$-2\sqrt{2}$	-4	$-2\sqrt{2}$	0	$2\sqrt{2}$	4

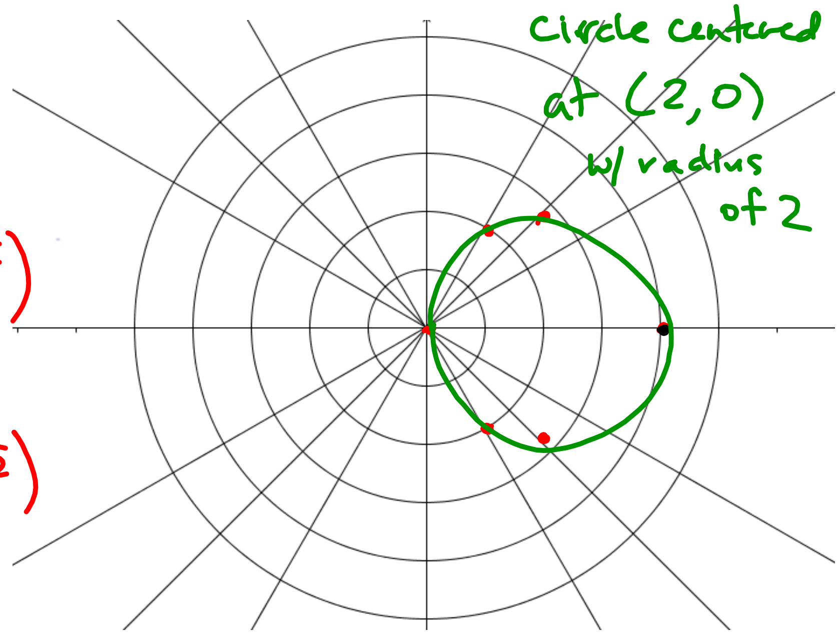
$4\left(\frac{\sqrt{2}}{2}\right)$

$(4,0) = (-4,\pi)$

$\left(2\sqrt{2}, \frac{\pi}{4}\right) = \left(-2\sqrt{2}, \frac{5\pi}{4}\right)$

$\left(0, \frac{\pi}{2}\right) = \left(0, \frac{3\pi}{2}\right)$

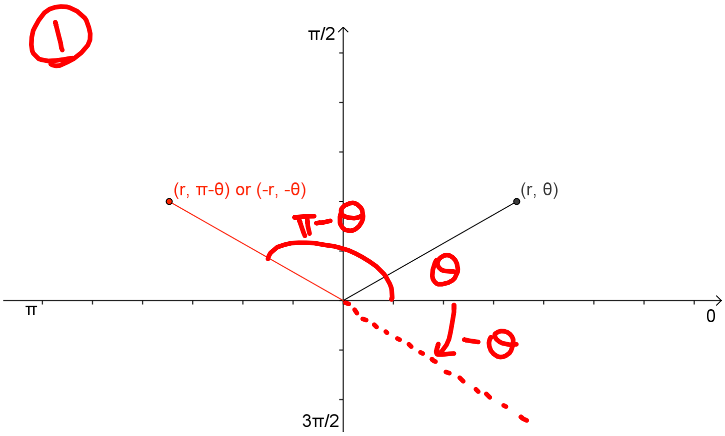
$\left(\frac{7\pi}{4}, 2\sqrt{2}\right) = \left(\frac{3\pi}{4}, -2\sqrt{2}\right)$



circle centered at (2,0) w/ radius of 2

Symmetry

1



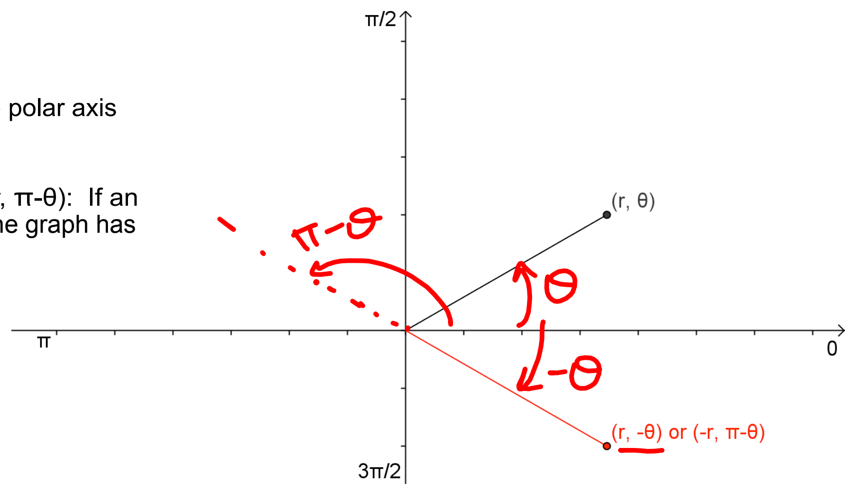
Symmetry with respect to the line θ=π/2 (y-axis)

Replace (r,θ) with (r,π-θ) or (-r, -θ): If an equivalent equation results, the graph has this type of symmetry.

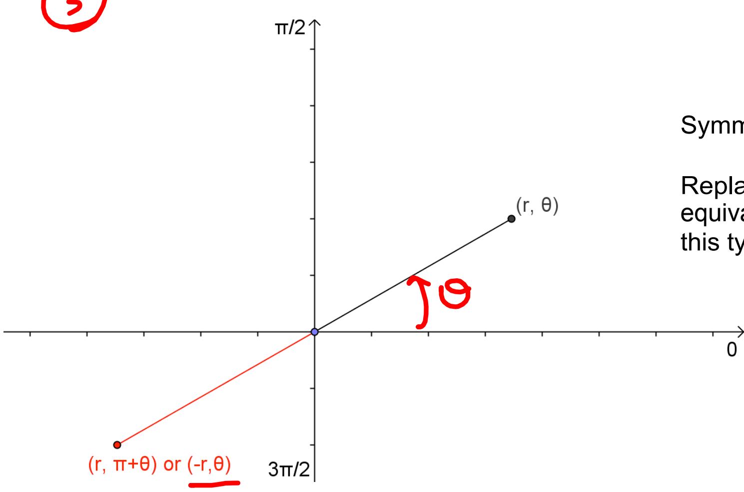
2

Symmetry with respect to the polar axis (θ=0): (x-axis)

Replace (r,θ) with (r,-θ) or (-r, π-θ): If an equivalent equation results, the graph has this type of symmetry.



3



Symmetry with respect to the pole (origin)

Replace (r,θ) with (-r,θ) or (r, π+θ): If an equivalent equation results, the graph has this type of symmetry.

If a polar equation passes a symmetry test, then its graph definitely exhibits that symmetry. However, if a polar equation fails a symmetry test, then its graph may or may not have that kind of symmetry.

Zeros and maximum *r*-values

Other helpful tools in graphing polar equations are knowing the values for θ for which $|r|$ is maximum and those for which $r = 0$.

Example 2 : Graph $r = \frac{1}{2} + \cos\theta$

Symmetry: ① replace (r,θ) w/ $(-r,-\theta)$: $-r = \frac{1}{2} + \cos(-\theta)$

$-r = \frac{1}{2} + \cos(\theta)$

$\not\Leftrightarrow\ r = \frac{1}{2} + \cos\theta$

② replace (r,θ) w/ $(r,-\theta)$: $r = \frac{1}{2} + \cos(-\theta)$

symmetry wrt $\theta = 0$ (x-axis) ✓ $r = \frac{1}{2} + \cos\theta$

③ replace (r,θ) w/ $(-r,\theta)$: $-r = \frac{1}{2} + \cos\theta$

$\not\Leftrightarrow\ r = \frac{1}{2} + \cos\theta$

$|r|$ maximum: $r = \frac{1}{2} + \cos\theta$

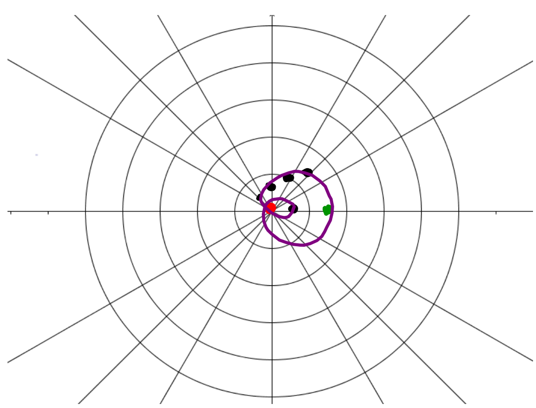
max when $\cos\theta = 1 \Rightarrow r$ max value $= 1\frac{1}{2} = \frac{3}{2}$ when $\theta = 0, 2\pi, ..$

Zero of r:

$0 = \frac{1}{2} + \cos\theta \Leftrightarrow \cos\theta = \frac{-1}{2} \Leftrightarrow \theta = \frac{2\pi}{3}, \frac{4\pi}{3}$

θ	0	$\pi/4$	$\pi/3$	$\pi/2$	$2\pi/3$	$3\pi/4$	π
r	$\frac{3}{2}$	$\frac{1+\sqrt{2}}{2}$	1	$\frac{1}{2}$	0	$\frac{1-\sqrt{2}}{2}$	$-\frac{1}{2}$

$r = \frac{1}{2} + \cos\theta$



Limaçon

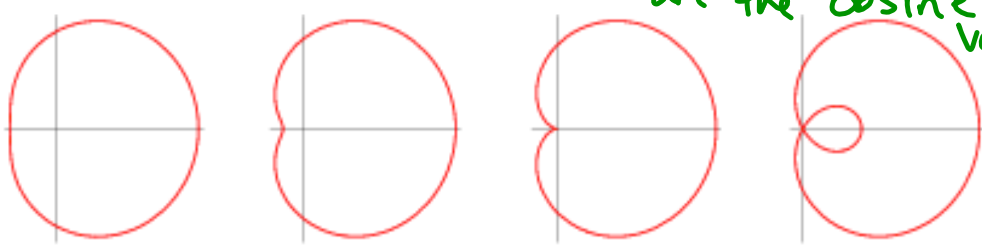
Limaçons

$$\boxed{\begin{aligned} r &= a \pm b\cos\theta \\ r &= a \pm b\sin\theta \end{aligned}}$$

a,b constants

$a > 0, \;\; b > 0$

(all these picture examples are the cosine varieties)



$\frac{a}{b} \geq 2$	$1 < \frac{a}{b} < 2$	$\frac{a}{b} = 1$	$\frac{a}{b} < 1$
Convex limaçon	Dimpled limaçon	Cardioid -always passes through pole	Limaçon with inner loop

Example 3: Graph $r = 3\sin 2\theta$

Symmetry: try replacing (r, θ) w/ $(-r, -\theta)$

$-r = 3\sin(-2\theta)$

$-r = -3\sin(2\theta)$

$r = 3\sin(2\theta)$ ✓

symmetry w.r.t $\theta = \frac{\pi}{2}$ (y-axis)

$|r|$ maximum:

r is max when $\sin(2\theta) = \pm 1 \Rightarrow 2\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}$

$(3, \frac{\pi}{4})$ $(-3, \frac{3\pi}{4})$

Zero of r: $0 = 3\sin(2\theta)$

$\sin(2\theta) = 0 \Rightarrow 2\theta = 0, \pi$

$\theta = 0, \frac{\pi}{2}$

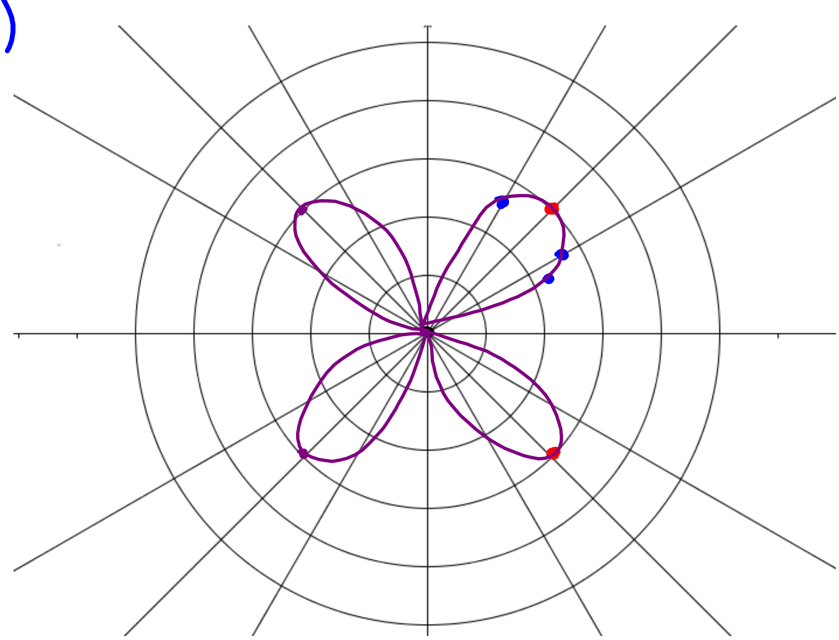
θ	0	$\pi/8$	$\pi/6$	$\pi/4$	$\pi/3$	$\pi/2$
r	0	$\frac{3\sqrt{2}}{2}$	$\frac{3\sqrt{3}}{2}$	3	$\frac{3\sqrt{3}}{2}$	0

$r = 3\sin(2\theta)$

$\frac{3\sqrt{2}}{2} \simeq 2.12$

$\frac{3\sqrt{3}}{2} \simeq 2.6$

$(-3, \frac{3\pi}{4})$



Roses

or

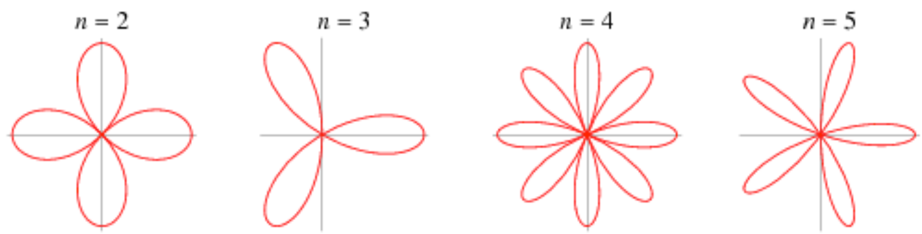
$$r = a \sin(n\,\theta),$$

$$r = a \cos(n\,\theta).$$

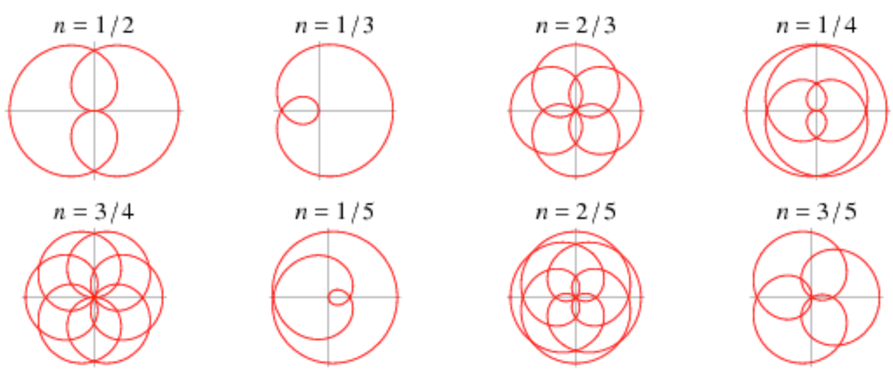
a constant

n constant

If n is odd, the rose is n-petalled. If n is even, the rose is $2\,n$-petalled.



No reason to limit ourselves to *n* integer:



Or even rational: