Department of Mathematics, University of Utah
Ph.D. Preliminary Examination: Applied Complex Variables and Asymptotic Methods
Spring 2024

This exam is closed book, closed notes, and no calculators are allowed. There is a formula sheet appended to this exam that you may use to complete the problems. You have two hours to complete this exam.

There are 5 problems below. You must complete 3 of them. Each problem is worth 20 points. Clearly indicate which 3 problems you wish to be graded; otherwise, the first 3 problems with work shown will be graded.

- A score of 52 (out of 60) is a *high pass.*
- A score of 48 (out of 60) is a *pass.*

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1. Complete the following:
   (a) (8 pts) Suppose that f is analytic in some domain with purely real values. Show that f is the constant function.
   (b) (12 pts) Consider two entire functions with no zeros and having a ratio equal to unity at infinity. Use Liouville's Theorem to show that they are in fact the same function. Justify all your steps, including satisfying the assumptions of Liouville's theorem.

2. (a) (10 pts) Compute the Laurent series expansion about the point $z = 0$ for,

$$f(z) = \frac{i+2}{(z+i)(z-2)},$$

in the region $1 < |z| < 2$.
   (b) (10 pts) Discuss the type of singularity (removable, pole and order, essential, branch, cluster). If the type is a pole give the strength of the pole, and give the nature (isolated or not). Include the point at infinity.

$$f(z) = \frac{e^{2z} - 1}{z}$$

3. Let f be an entire function, with $|f(z)| \leq C|z|^2$ for all z, where C is a constant. Show that $f(z) = az^2 + bz$ for some constants a, b.

4. Show that,

$$\int_0^\infty \frac{\cos 3x}{(x^2+1)^2} \mathrm{d}x = \frac{\pi}{e^3}$$

Clearly indicate the contour of integration and the treatment of the integral along each segment.

5. Compute the complete asymptotic expansion in powers of k of,

$$I(k) = \int_0^\infty e^{-kt^2} \frac{\sin(t^4)}{t^{7/2}} \mathrm{d}t, \qquad k \to \infty$$

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Formula sheet

The Euler Gamma function for real inputs is defined as,

$$\Gamma(x) := \int_0^\infty t^{x-1} e^{-t} \mathrm{d}t, \qquad x > 0.$$

In what follows, C_R denotes a semicircular arc of radius R in the upper half-plane centered at the origin. The contour C_ϵ is a circular arc of radius ϵ centered around a point z_0 that sweeps out an angle of ϕ.

1. Suppose f is analytic on an open domain containing a simple closed loop C. Then for all integers $n \geq 0$ and all z enclosed by C,

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(w)}{(w-z)^{n+1}} \mathrm{d}w,$$

2. The coefficients for a Laurent series of the function f are given by,

$$c_n = \frac{1}{2\pi i} \oint_C \frac{f(w)}{(w-z)^{n+1}} \mathrm{d}w$$

3. If a continuous f is bounded over a contour C of finite length, i.e., $|f(z)| \leq M < \infty$ for all $z \in C$ and $\int_C |\mathrm{d}z| = L < \infty$, then

$$\left| \int_C f(z) \mathrm{d}z \right| \leq ML$$

4. Suppose $f(z) = P(z)/Q(z)$ is a rational function with $\deg Q \geq \deg P + 2$. Then,

$$\lim_{R\to\infty} \int_{C_R} f(z) \mathrm{d}z = 0.$$

5. (Jordan's Lemma) Suppose that $f(z) \to 0$ uniformly for $z \in C_R$ as $R \to \infty$. Then for any $k > 0$,

$$\lim_{R\to\infty} \int_{C_R} e^{ikz} f(z) \mathrm{d}z = 0.$$

6. Suppose that $(z - z_0) f(z) \to 0$ uniformly for $z \in C_\epsilon$ as $\epsilon \to 0$. Then,

$$\lim_{\epsilon\to 0} \int_{C_\epsilon} f(z) \mathrm{d}z = 0.$$

7. Suppose that f has a simple pole at $z = z_0$. Then

$$\lim_{\epsilon\to 0} \int_{C_\epsilon} f(z) \mathrm{d}z = i\phi \mathrm{Res}(f; z_0).$$

8. With C_R any origin-centered circular arc (not necessarily in the upper half-plane), if $zf(z) \to 0$ uniformly on C_R as $R \to 0$, then,

$$\lim_{R\to\infty} \int_{C_R} f(z) \mathrm{d}z = 0.$$

Laplace-type integrals

These are formulas regarding asymptotic ($k \to \infty$) behavior of $I(k) := \int_a^b f(t)e^{-k\phi(t)}\mathrm{d}t$ for $a < b$.

(1) (Watson's Lemma) Set $a = 0$ and $\phi(t) = t$. Assume f is integrable with the series expansion,

$$f(t) \sim t^\alpha \sum_{n=0}^{\infty} a_n t^{\beta n} \quad t \to 0^+, \quad \alpha > -1, \quad \beta > 0.$$

In addition, if $b < \infty$ then assume $|f(t)| \leq M < \infty$ for $t \in [a, b]$, and if $b = \infty$ then assume $f(t) = \mathcal{O}(e^{ct})$ as $t \to \infty$ for some $c \in \mathbb{R}$. Then,

$$I(k) \sim \sum_{n=0}^{\infty} a_n \frac{\Gamma(\alpha + \beta n + 1)}{k^{\alpha+\beta n+1}}.$$

(2) (Laplace's Method) Assume $b < \infty$, and that $\phi \in C^4([a, b])$ and $f \in C^2([a, b])$. Suppose that for some $c \in [a, b]$, we have $\phi'(c) = 0$ and $\phi''(c) > 0$. Also, assume that $\phi'(t) \neq 0$ for all $t \in [a, b]\backslash\{c\}$. Then,

$$I(k) \sim G(c)e^{-k\phi(c)} f(c)\sqrt{\frac{2\pi}{k\phi''(c)}} + \mathcal{O}\left(\frac{e^{-k\phi(c)}}{k^{G(c)+1/2}}\right), \qquad G(c) := \begin{cases} 1, & c \in (a, b) \\ \frac{1}{2}, & \text{otherwise} \end{cases}$$

Fourier-type integrals

These are formulas regarding asymptotic ($k \to \infty$) behavior of $I(k) := \int_a^b f(t)e^{ik\phi(t)}\mathrm{d}t$ for $a < b$.

(1) Set $a = 0$, and $\phi(t) = \mu t$, where $\mu = \pm 1$, and $k > 0$. Suppose f vanishes infinitely smoothly at $t = b$, that $f \in C^\infty((0, b])$, and that for some $\gamma > -1$, $f(t) \sim t^\gamma + o(t^\gamma)$ as $t \to 0^+$. Then,

$$I(k) = \left(\frac{1}{k}\right)^{\gamma+1} \Gamma(\gamma + 1)e^{i\frac{\pi}{2}\mu(\gamma+1)} + o(k^{-(\gamma+1)}).$$

(2) (Stationary phase) Suppose $c \in (a, b)$ is the only value of t where $\phi'(t)$ vanishes. Assume that f vanishes infinitely smoothly at both $t = a$ and $t = b$, and that both f and ϕ are C^∞ on the intervals $[a, c)$ and $(c, b]$. Suppose that there is some $\gamma > -1$ such that as $t \to c$,

$$\begin{aligned} \phi(t) - \phi(c) &\sim \alpha(t - c)^2 + o((t - c)^2), \\ f(t) &\sim \beta(t - c)^\gamma + o((t - c)^\gamma). \end{aligned}$$

Then with $\mu = \operatorname{sgn} \alpha$,

$$\int_a^b f(t)e^{ik\phi(t)}\mathrm{d}t \sim e^{ik\phi(c)}\beta\Gamma\left(\frac{\gamma + 1}{2}\right) e^{i\pi\frac{\gamma+1}{4}\mu} \left(\frac{1}{k|\alpha|}\right)^{\frac{\gamma+1}{2}} + o\left(k^{-\frac{\gamma+1}{2}}\right).$$