UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS
Ph.D. Preliminary Examination in Complex Analysis
May 26, 2023

Instructions. Answer as many questions as you can. Each question is worth 10 points. For a high pass you need to solve *completely* at least three problems and score at least 30 points. For a pass you need to solve *completely* at least two problems and score at least 25 points. Carefully state any theorems you use.

Notation: $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$

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1. What is the group of all biholomorphisms of $\mathbb{C} \setminus \{0\}$?

2. Let $f : \mathbb{D} \to \mathbb{D}$ be holomorphic and $f(0) = a$. Show that f has no zeros in $\{z \in \mathbb{C} \mid |z| < |a|\}$.

3. Prove the following version of Morera's theorem. If $f : \mathbb{D} \to \mathbb{C}$ is a continuous function such that for every rectangle $R \subset \mathbb{D}$

$$\int_{\partial R} f(z)dz = 0$$

then f is holomorphic. Note: It does not follow from the assumption in any obvious way that integrals over triangles are 0.

4. Find all singularities of the function $\frac{1}{e^z+1}$ in $\mathbb{C}$. If they are poles, compute the residue.

5. Let f be holomorphic on $\mathbb{C} \setminus \{0, 2\}$ with simple poles at 0 and 2. Suppose that $\int_{|z|=1} f(z)dz = 2\pi i$ and $\int_{|z|=3} f(z)dz = 0$. Also suppose that f is bounded on $|z| > 3$. List all such functions (with a proof that the list is complete).

6. Suppose a sequence of holomorphic functions $f_n : \mathbb{D} \to \mathbb{C}$ converges pointwise to a function $f : \mathbb{D} \to \mathbb{C}$, and that all f_n are uniformly bounded. Show that convergence is uniform on every disk

$$D_r = \{z \in \mathbb{C} \mid |z| \leq r\}$$

for $r < 1$.