UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS

Ph.D. preliminary Examination on
Applied Linear Operators and Spectral Methods (Math 6710)
January 2024

Instructions: This examination includes five problems but you are to work three of them. If you work more than the required number of problems, then state which problems you wish to be graded, otherwise the first three problems will be graded. All problems are worth 20 points. A pass is 35 or more points. A high-pass is 45 or more points.

1. Let (λ_j) be a sequence that is dense in $[0,1]$ and consider the linear operator $T : \ell^2 \to \ell^2$ defined by its action on a sequence $(\xi_j) \in \ell^2$ by

$$T(\xi_j) = (\lambda_j \xi_j).$$

   Find the spectrum $\sigma(T)$, point spectrum $\sigma_p(T)$, continuum spectrum $\sigma_c(T)$, residual spectrum $\sigma_r(T)$ and resolvent $\rho(T)$.

2. Let H be a Hilbert space and $T : H \to H$ be a compact operator. Show that the Hilbert adjoint T^* is also compact.

3. Let X be a normed vector space. Use the Hahn-Banach theorem to show that for any $x \in X$ we have that

$$||x|| = \sup_{f \in X', f \neq 0} \frac{|f(x)|}{||f||}.$$

4. Let M be a subset of a Hilbert space H. Assume M is such that for any $v, w \in H$ for which the equality

$$\langle v, x \rangle = \langle w, x \rangle$$

   holds for all $x \in M$, we must have $v = w$. Show that $M^{\perp} = \{0\}$.

5. Let X be a Banach space, $f : X \to \mathbb{R}$ be a bounded linear functional, $y \in X$ be fixed, and $\alpha \in \mathbb{R}$.

   (a) Prove that there exists a constant $C > 0$ such that if $|\alpha| < C$, then the non-linear equation

$$x + \alpha f(x) x = y \qquad (1)$$

   has a unique solution x in the ball $B = \{x \in X : ||x - y|| \leq 1\}$.

   (b) Suggest an interative procedure for approximating the unique solution x to equation (1).