UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS

Ph.D. Preliminary Examination in Ordinary Differential Equations

January 5, 2024.

Instructions: This examination consists of seven problems. You are to work four problems. If you work more than the required number of problems, then state which problems you wish to be graded, otherwise the first four will be graded.

In order to receive maximum credit, solutions to problems must be clearly and carefully presented and should be detailed as possible. All problems are worth [25] points. A high-passing score is [75] and a passing score is [60].

Do four problems for full credit

1. Let $f(t,x) : [0,\infty) \times \mathbf{R}^d \to \mathbf{R}^d$ be a continuous function and $x_0 \in \mathbf{R}^d$. Suppose that there are constants L, $M < \infty$ such that

$$|f(t,y) - f(t,z)| \le L|y-z| \qquad \text{and} \qquad |f(t,0)| \le M \qquad \text{for all } t,\ y,\ z.$$

Show that the initial value problem

$$\begin{aligned} \dot{x} &= f(t,x) \\ x(0) &= x_0 \end{aligned}$$

has a unique solution $x(t) \in \mathcal{C}^1([0,\infty), \mathbf{R}^d)$. [Don't just quote a result. State the theorem and give as complete and detailed a proof as you can.]

2. (a) Let A be an $d \times d$ real matrix. State and prove a theorem that gives necessary and sufficient conditions on A so that every solution $x(t)$ is bounded for $t \in [0,\infty)$.

$$\dot{x} = Ax. \tag{1}$$

(b) Let $A(t)$ be a $\mathcal{C}^1$, $T > 0$ periodic, $d \times d$ real continuous matrix function. State but don't prove a theorem that gives necessary and sufficient conditions on $A(t)$ so that every solution $x(t)$ is bounded for $t \in [0,\infty)$.

$$\dot{x} = A(t)\, x.$$

3. Let $B(t)$ be a $\mathcal{C}^1$ real $d \times d$ matrix function defined for $t \in [0,\infty)$ such that

$$\lim_{t\to\infty} B(t) = 0.$$

Let A be a real $d \times d$ matrix such that $\Re e\, \lambda < 0$ for all eigenvalues of A. Show that there is $\delta > 0$ such that if $|x_0| \le \delta$, then the initial value problem

$$\begin{aligned} \dot{x} &= Ax + B(t)x \\ x(0) &= x_0 \end{aligned} \tag{2}$$

has a solution such that $x(t) \to 0$ as $t \to \infty$. [Provide the complete proof. Do not just quote a theorem.]

[Hint: Use the variation of parameters formula and Gronwall's Inequality, which says that for $f(t)$, $\varphi(t)$ nonnegative continuous functions on the interval $[0, T]$ and for $c_0 \geq 0$ if $f(t) \leq c_0 + \int_0^t \varphi(s) f(s)\, ds$ for all $t \in [0, T]$ then $f(t) \leq c_0 \exp\left(\int_0^t \varphi(s)\, ds\right)$ for all $t \in [0, T]$. You might first show that there is a constant k such that if ϵ is small enough and $|x_0| < \epsilon$ then the solution remains bounded $|x(t)| \leq k|x_0|$ for all $t \in [0, T]$.]

4. Prove that there exists a nonconstant periodic orbit.

$$\begin{aligned} \dot{x} &= y \\ \dot{y} &= -x + y(1 - x^2 - 2y^2). \end{aligned}$$

5. Determine whether the given equilibrium point for the given system is Liapunov stable, is attractive, is asymptotically stable, or is not stable. Give a brief explanation.

   (a) $(x, y, z) = (0, 0, 0)$ for $\begin{cases} \dot{x} = y - x \\ \dot{y} = -y - xz \\ \dot{z} = xy - z \end{cases}$

   (b) $(x, y) = (0, 0)$ for $\begin{cases} \dot{x} = -\dfrac{\partial V}{\partial x}(x, y) \\ \dot{y} = -\dfrac{\partial V}{\partial y}(x, y) \end{cases}$ where $V(x, y) = x^4 - x^2 + y^2$

   (c) The zero solution $z(t) = 0$ for $\ddot{x} + \dot{x} + g(x) = 0$ where $g \in \mathcal{C}^1(\mathbf{R})$ and $xg(x) > 0$ for all $x \neq 0$.

6. Find the approximate center manifold at the origin. Use it to determine the stability type of the origin in the system

$$\begin{aligned} \dot{x} &= xy \\ \dot{y} &= -y + x^2 \end{aligned}$$

7. Let A be a real $d \times d$ matrix such that $\Re e\, \lambda < 0$ for all eigenvalues of A. Consider the system

$$\dot{x} = Ax + h(t) \tag{3}$$

where $h(t) \in \mathbf{R}^d$ is a T-periodic continuous function.

   (a) Define the *Poincaré Map* in this context and give an explicit formula for the Poincaré Map in terms of A and h

   (b) Show that there is a unique T-periodic solution of (3). [Hint: there is again an explicit equation in terms of A and h.]

   (c) What stability property does the periodic orbit have? Provide a justification of your answer with explicit calculation/proof.

   (d) How could the requirements stipulated on the eigenvalues of A be generalized to still maintain the existence and uniqueness results for the periodic orbit from (b)?