Qualifying Exam
Analysis of Numerical Methods I, January 2022

Instructions: This exam is closed book, no notes, and no electronic devices or calculators are allowed. You have two hours and you will be graded on work for only 3 out of the 4 questions below. All questions have equal weight and a cumulative score of 65% or more on your 3 graded questions is considered a pass. A cumulative score of 80% or greater is considered a high pass. Indicate clearly the work and the 3 questions that you wish to be graded.

Problem 1. (**Rank-One Perturbation of the Identity**).
If u and v are n-vectors, the matrix $B = I + uv^\star$ is known as a *rank-one perturbation of the identity*. Show that if B is nonsingular, then its inverse has the form $B^{-1} = I + \beta uv^\star$ for some scalar β, and give an expression for β.
For what u and v is B singular? If it is singular, what is $null(B)$?

Problem 2. (**Properties via SVD**).
a) Consider $A \in \mathrm{C}^{m\times n}$. Define what we mean by the singular value decomposition of A.
b) Show that the rank of A is r, the number of nonzero singular values.
c) Show that the largest singular value of A, call it here $\sigma_{max}(A)$, satisfies the relation

$$\sigma_{max}(A) = \max_{y\in\mathbb{C}^m, x\in\mathbb{C}^n} \frac{|y^\star Ax|}{||x||_2||y||_2}.$$

Problem 3. (**Properties of Projectors**).
Prove algebraically:
a) Show that if $P \in \mathrm{C}^{m\times m}$ is a nonzero projector, then $||P||_2 \geq 1$.
b) Show that if P is an orthogonal projector, then $I - 2P$ is unitary.

Problem 4. (**Hadamard's Inequality**).
Prove algebraically:
Let $A \in \mathrm{C}^{m\times m}$ and let $\mathbf{a}_j$ denote the j^{th} column of A. Then, show that,

$$|det(A)| \leq \mathbf{\Pi}_{j=1}^{m} ||\mathbf{a}_j||_2.$$