Qualifying Exam
Analysis of Numerical Methods I, August 2024

Instructions: This exam is closed book, no notes, and no electronic devices or calculators are allowed. You have two hours and you will be graded on work for only 3 out of the 4 questions below. All questions have equal weight and a cumulative score of 65% or more on your 3 graded questions is considered a pass. A cumulative score of 80% or greater is considered a high pass. Indicate clearly the work and the 3 questions that you wish to be graded.

Problem 1.(**Matrix Norms**).
Show that $||A||_1 = \max\limits_{j=1,\ldots,n} \sum_{i=1}^{n} |a_{ij}|$.

Problem 2. (**Properties via SVD**).
Suppose $A \in \mathrm{C}^{m\times n}$ has an SVD, $A = U\Sigma V^*$, where u_j are the left singular vectors, $U = \{u_j\}$, v_j are the right singular vectors, $V = \{v_j\}$, and σ_j are the singular values. Assume rank of the matrix A is r. For any ν with $0 \le \nu \le r$, define

$$A_\nu = \sum_{j=1}^{\nu} \sigma_j u_j v_j^\star,$$

(if $\nu = p = min\{m, n\}$, define $\sigma_{\nu+1} = 0$). Show that,

$$||A - A_\nu||_2 = \inf_{\substack{B\in \mathrm{C}^{m\times n} \\ rank(B)\le\nu}} ||A - B||_2 = \sigma_{\nu+1}.$$

Problem 3. (**Eigenvalues and Eigenvectors**).
a) State the Rayleigh Quotient Iteration Algorithm to find eigenvalues/eigenvectors for general real symmetric matrix $A \in \mathrm{R}^{m\times m}$.
b) Prove that the spectrum (the set of all eigenvalues) of any matrix $A \in \mathrm{C}^{m\times m}$ is contained in the union of the following m disks, D_i, in the complex plane:

$$D_i = \{z \in \mathrm{C} : |z - a_{ii}| \le \sum_{j=1, j\ne i}^{m} |a_{ij}|\}, \quad 1 \le i \le m$$

Problem 4. (**Properties of the SPD Matrices**)
Let $A \in \mathrm{R}^{n\times n}$ be a symmetric positive definite matrix (spd). Show that there exists a unique upper triangular matrix $H \in \mathrm{R}^{n\times n}$ with positive diagonal entries such that, $A = H^T H$.