UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS
Ph.D. Preliminary Examination in Partial Differential Equations
August 15/16, 2024.

Instructions: There are six total problems. Problems will be scored out of 10 points and four (4) problems will be graded. Clearly identify which four (4) problems you want to be graded. Partial credit will be given for *significant* progress towards the solution.

Three (3) completely correct problems will be a High Pass and two (2) completely correct problems with sufficient partial credit for at least 26 total points will be a Pass. Be sure to provide all relevant definitions and statements of theorems cited. All solutions must include rigorous justification unless otherwise indicated.

Problem 1. State (carefully) and prove the standard L^2 Poincaré inequality in a smooth bounded domain U.

Problem 2. Let U be a smooth bounded domain and $f : \bar{U} \to \mathbb{R}$ be a smooth function. Consider the energy

$$I[v] := \int_U \sqrt{1 + |\nabla v|^2} - fv \; dx$$

on the admissible class

$$\mathcal{A} = \{v \in C^2(U) \cap C^1(\overline{U}) : \; v = 0 \;\text{ on }\; \partial U\}.$$

Suppose that there is $u \in \mathcal{A}$ satisfying

$$I[u] = \min_{v \in \mathcal{A}} I[v].$$

Find and justify the PDE boundary value problem satisfied by u.

Problem 3. Let U a bounded domain in $\mathbb{R}^n$. Consider a smooth solution u of the equation

$$\begin{cases} u_t = \Delta u + c(x,t)u & \text{in } U \times (0,\infty), \\ u = 0 & \text{on } \partial U \times [0,\infty), \\ u(x,0) = f(x). \end{cases}$$

Show that, if $c(x,t) \leq -\gamma$ for all x and t, then

$$\max_{x \in \overline{U}} |u(x,t)| \leq e^{-\gamma t} \max_{\overline{U}} |f|.$$

Problem 4. Consider the scalar conservation law,

$$u_t + (\tfrac{1}{4}u^4)_x = 0 \;\text{ in }\; \mathbb{R} \times (0,\infty).$$

Find the entropy satisfying weak solution for the initial data,

$$u(x,0) = \begin{cases} 0 & x < 0 \text{ or } x > 1 \\ 1 & 0 < x < 1 \end{cases}$$

Make a drawing of the characteristics in the (x,t)-plane including any shocks and rarefactions.

Problem 5. Use the method of characteristics to solve

$$xu_x + yu_y = u \quad \text{in} \quad \{y > 0\}$$

with

$$u(x,1) = \frac{1}{1+x^2}.$$

Problem 6. Consider the wave equation in a two-dimensional quadrant

$$u_{tt} - \Delta u = 0 \text{ in } \{x > 0, y > 0\} \times (0, \infty)$$

with boundary conditions

$$\partial_x u(0,y,t) = 0 \quad \text{and} \quad u(x,0,t) = 0$$

and initial condition u_0, supported in the ball

$$B_1((2,2)) = \{(x,y) : (x-2)^2 + (y-2)^2 \leq 1\}.$$

Describe, precisely and with careful justification, the support of $u(x,y,t)$ at all positive times $t > 0$. **Note:** The support of a function $f(x,y)$ is the closure of the set $\{(x,y) : f(x,y) \neq 0\}$.