University of Utah, Department of Mathematics
August 2023, Algebra I Qualifying Exam

There are five problems on the exam. You may attempt as many problems as you wish; three correct solutions count as a high pass. 2 correct solutions count as a pass. Show all your work, and provide reasonable justification for your answers.

1. Let k be the field with 2 elements. Identify all the maximal ideals in the ring $R = k[x,y]/(xy, x^2+y+1)$

2. Suppose R is a commutative ring with 1 and $x, y \in R$ satisfy $xy = 0$ and $(x,y) = R$ (i.e. the ideal generated by x and y is the whole ring). Prove that the R-module homomorphism

$$\psi : R \longrightarrow R/(x) \oplus R/(y)$$

sending $f \in R$ to its image in $R/(x)$, and in $R/(y)$, is an isomorphism.

3. Suppose $R = \mathbb{Q}[x,y]$ is a polynomial ring. Compute the $\mathbb{Q}$-vector space dimension of

$$\mathrm{Tor}_1^R\Big(\mathrm{Ext}_R^1\big(R/(x^2), R/(y^2)\big), R/(y)\Big)$$

4. If some power of a linear map $A : \mathbb{C}^n \to \mathbb{C}^n$ is the identity (i.e. A has finite order), show that A is diagonalizable, i.e. there is a change of basis matrix B such that BAB^{-1} is diagonal.

5. Find four non-isomorphic $\mathbb{Z}$-modules (i.e. abelian groups) M that each fit into an exact sequence of the form:

$$0 \to \mathbb{Z} \to M \to \mathbb{Z}/6\mathbb{Z} \to 0$$