UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS
Ph.D. Preliminary Examination in Real Analysis
August, 2023.

Instructions. Answer as many questions as you can. Each question is worth 10 points. For a high pass you need to solve *completely* at least three problems and score at least 30 points. For a pass you need to solve *completely* at least two problems and score at least 25 points. Carefully state any theorems you use.

Let λ denote Lebesgue measure on $\mathbb{R}$.

1. Is $\{v \in \ell^2(\mathbb{N}) : |v_i| \le \frac{1}{\log(i+4)}\}$ compact in the norm topology on $\ell^2(\mathbb{N})$? Justify your answer.

2. Let $f : \mathbb{R} \to \mathbb{C}$ be Lebesgue measurable. Assume that f is invariant under translations by rational numbers. Show that f is constant Lebesgue almost everywhere.

3. Define the Fourier transform on $\mathbb{R}$ (do not worry about the choice of normalization). Show that the Fourier transform of an $L^1(\mathbb{R}, \lambda)$ function is in $C_0(\mathbb{R})$. You may use that the Fourier transform of a continuous compactly supported function is in $C_0(\mathbb{R})$.

4. Let $X = [0,1]$ and let μ be a finite Borel measure on X. Prove that μ is *regular*, i.e. that for any Borel set $A \subseteq X$ the following holds:

   (i) $\mu(A) = \sup\{\mu(K) \mid K \subseteq A, K \text{ compact}\}$, and

   (ii) $\mu(A) = \inf\{\mu(U) \mid U \supseteq A, U \text{ open}\}$.

   Note: The same statement holds for any compact metrizable space X.

5. Let f be Lebesgue integrable on $(0,1)$. For $x \in (0,1)$ define

$$g(x) = \int_x^1 \frac{f(t)}{t} d\lambda.$$

   Prove that g is Lebesgue integrable on $(0,1)$ and that

$$\int_0^1 g(x) d\lambda = \int_0^1 f(x) d\lambda.$$

6. Let $(X, \mathcal{B}, \mu)$ be a finite measure space and $f : X \to \mathbb{R}$ be measurable.

(a) Show that if $g_n(x) = (f(x))^n$ and $\int g_n d\mu$ is uniformly bounded for all n then $|f(x)| \leq 1$ (μ-a.e).

(b) Show that $\int_X g_n d\mu$ is independent of n iff there is $A \in \mathcal{B}$ so that $f = \chi_A$ (μ a.e.).