University of Utah, Department of Mathematics
Algebra 1 Qualifying Exam
August 2021

There are five problems on this exam. You may attempt as many problems as you wish; two correct solutions count as a *pass*, and three correct as a *high pass*. Show all your work, and provide reasonable justification for your answers.

1. Consider the ring homomorphism given by

$$\begin{aligned} \mathbb{Z}[x] &\longrightarrow \quad \mathbb{Q} \\ f(x) &\longmapsto f(1/2) \end{aligned}$$

   Determine a minimal generating set for the kernel.

2. Let $R := \mathbb{Q}[x]$, and let M be the cokernel of the map

$$R^3 \xrightarrow{\begin{pmatrix} x-2 & 4 & x \\ 0 & 4 & x \\ 0 & -x & -1 \end{pmatrix}} R^3.$$

   Write M as a direct sum of cyclic R-modules.

3. Pro ve that $\mathbb{C}[x]$ is integrally closed in $\mathbb{C}(x)$, the field of rational functions.

4. Let I be the ideal $(2,\ 1+\sqrt{-5})$ in the ring $R := \mathbb{Z}[\sqrt{-5}]$, and let $\Lambda^2(I)$ denote the second exterior power of I in the category of R-modules. Prove or disprove: $\Lambda^2(I)$ is zero.

5. Let R be a commutative ring. Let P and Q be two projective R-modules. Prove that $P \otimes_R Q$ is projective.