Department of Mathematics
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Real and Complex Analysis Preliminary Examination

January 7, 2011

Instructions: Do seven problems and list on the front of your blue book the seven problems to be graded. Do at least three problems from each part.

Part A:

Problem 1. Recall that a topological space is said to be separable if it contains a countable dense set. Also recall that $\ell^p(\mathbb{Z})$ is the L^p-space for counting measure on the integers. Prove or disprove the following statements:

(a) $\ell^1(\mathbb{Z})$ is separable.
(b) $\ell^\infty(\mathbb{Z})$ is separable.

Problem 2. Let f be a real-valued function on a measure space X. Prove or disprove the following statements:

(a) If f is measurable, then so is $|f|$.
(b) If $|f|$ is measurable, then so is f.

Problem 3. Let f be a continuous function on the circle $\{e^{i\theta} \mid 0 \le \theta < 2\pi\}$. Let

$$c_n := \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) e^{-in\theta} d\theta.$$

Prove or disprove:

$$f(e^{i\theta}) = \lim_{N \to \infty} \left(\sum_{n=-N}^{N} c_n e^{in\theta} \right)$$

for all θ.

Problem 4. Let $\mathbb{R}$ be equipped with Lebesque measure. Suppose $f \in L^1(\mathbb{R})$ is uniformly continuous. Prove that $\lim_{|x| \to \infty} f(x) = 0$.

Problem 5. Let (X, μ) be a measure space, let $\{f_n\}$ be a sequence of non-negative integrable functions, and assume there exists an integrable function f such that $f(x) = \lim_{n \to \infty} f_n(x)$ almost everywhere. Further assume

$$\lim_{n \to \infty} \int_X f_n d\mu = \int_X f d\mu. \tag{1}$$

Prove that

$$\lim_{n \to \infty} \int_X |f_n - f| d\mu = 0$$

Show by example that this need not hold if the assumption (1) is omitted.

Part B:

Problem 6. Let f be holomorphic function defined on a domain $U \subset \mathbb{C}$. Let $\bar{U} = \{z \in \mathbb{C} \mid \bar{z} \in U\}$. Consider the function $g(z) = \overline{f(\bar{z})}$ on $\bar{U}$. Is g holomorphic or not? Explain your answer!

Problem 7. Let f be a meromorphic function on $\mathbb{C}$ such that there exist $R, C > 0$ and positive integer k such that $|f(z)| \leq C|z|^k$ for all $|z| \geq R$. Prove that f is a rational function.

Problem 8. Let f be a holomorphic function on $U \subset \mathbb{C}$. Let $a \in U$ and k a positive integer. Prove that the following statements are equivalent:

(i) a is a zero of order $\geq k$ of f;

(ii) there exist $C, \epsilon > 0$ such that

$$|f(z)| \leq C|z-a|^k$$

for all z such that $|z-a| < \epsilon$.

Problem 9. Find all isolated singularities of the function

$$f(z) = \sin(z)\sin\left(\frac{1}{z}\right)$$

in $\mathbb{C}$. Determine the residues of f at these isolated singularities.

Problem 10. Evaluate the integral

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^2 + 2x + 10}\, dx$$

using the residue theorem.