Probability Qualifying Examination

August 18, 2009

There are 11 problems, of which you must turn in solutions for **exactly** 6 (your best 6, in your opinion). Each problem is worth 10 points, and 40 points is required for passing. On the outside of your exam book, indicate which 6 you have attempted.

If you think a problem is misstated, interpret it in such a way as to make it nontrivial.

1. Let X be an exponential random variable with mean 1, that is, $P(X > u) = e^{-u}$ for all $u > 0$. Evaluate $E[X \mid X \wedge t]$ and $E[X \mid X \vee t]$ for all $t > 0$. Here $a \wedge b := \min(a, b)$ and $a \vee b := \max(a, b)$.

2. Let X be a random variable with values in an interval I, and suppose that f and g are nondecreasing functions on I such that $f(X)$ and $g(X)$ have finite variance. Show that $\mathrm{Cov}(f(X), g(X)) \geq 0$.

   Hint: If X_1 and X_2 are i.i.d. as X, then $(f(X_1)-f(X_2))(g(X_1)-g(X_2)) \geq 0$.

3. Let $X_1, X_2, \ldots$ be independent with

   $$\mathrm{P}(X_n = n^2 - 1) = n^{-2} = 1 - \mathrm{P}(X_n = -1), \qquad n \geq 1.$$

   Notice that $\mathrm{E}[X_n] = 0$ for all $n \geq 1$, which might lead one to expect that $S_n := X_1 + \cdots + X_n$ satisfies $S_n/n \to 0$ a.s. Show that in fact $S_n/n \to -1$ a.s.

4. Let $X_1, X_2, \ldots$ be an i.i.d. sequence with $\mathrm{P}(X_1 = 1) = p$ and $\mathrm{P}(X_1 = 0) = 1 - p$, where $0 < p < 1$. For each $m \geq 1$, define

   $$N_m := \min\{n \geq m : (X_{n-m+1}, \ldots, X_n) = (1, 1, \ldots, 1)\}$$

   and show that

   $$\mathrm{E}[N_m] = \frac{1}{p} + \frac{1}{p^2} + \cdots + \frac{1}{p^m}.$$

   Hint: Evaluate $\mathrm{E}[N_m]$ by conditioning on N_{m-1}.

5. Consider a deck of n cards labeled $1, 2, \ldots, n$. Assume it is well shuffled with every possible arrangement equally likely. Let E_n be the event that card j is in position j in the shuffled deck for some $j \in \{1, 2, \ldots, n\}$. Evaluate $P(E_n)$ using inclusion-exclusion and show that $\lim_{n\to\infty} P(E_n) = 1 - e^{-1}$.

6. The Cauchy distribution has density $f(x) := 1/[\pi(1 + x^2)]$ for $-\infty < x < \infty$ and characteristic function $\varphi(t) := e^{-|t|}$ for $-\infty < t < \infty$. Let $X_1, X_2, \ldots$ be i.i.d. Cauchy, put $S_n := X_1 + \cdots + X_n$ for each $n \geq 1$. Show that S_n/n does not converge almost surely or in probability, but does converge in distribution.

7. Let $X_1, X_2, \ldots$ be i.i.d. uniform$(0\,, 2)$, and put $M_n := X_1 X_2 \cdots X_n$ for each $n \geq 1$. Show that $\{M_n\}_{n=1}^{\infty}$ is a martingale. What does the martingale convergence theorem tell us about $\lim_{n\to\infty} M_n$? (In particular, evaluate this limit.)

8. Suppose X_1 and X_2 are independent random variables that satisfy the following three properties: (i) $Y_1 := (X_1 + X_2)/\sqrt{2}$ is standard normal; (ii) $Y_2 := (X_1 - X_2)/\sqrt{2}$ is standard normal; and (iii) Y_1 and Y_2 are independent. Prove that X_1 and X_2 are normally distributed. Can you compute their respective means and variances?

9. Prove that if $P\{X \geq 0\} = 1$ and $0 < E(X^2) < \infty$, then

$$P\{X = 0\} \leq \frac{\operatorname{Var} X}{E(X^2)}.$$

10. Let X and Y be independent, standard-normal random variables. Find the distribution of X/Y?

11. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. random variables, each taking the values 0 and 1 with equal probabilities 1/2. Define

$$U := \sum_{n=1}^{\infty} \frac{X_n}{2^n}.$$

Prove that U is distributed uniformly on $(0\,, a)$ and compute a.