Probability Prelim Exam

August 2017

Instructions (Read before you begin)

- You may attempt all of 10 problems in this exam. However, you can turn in solutions for **at most** 6 problems. On the outside of your exam booklet, indicate which problem you are turning in.
- Each problem is worth 10 points; 40 points or higher will result in a pass.
- If you think that a problem is misstated, then interpret that problem in such a way as to make it nontrivial.
- If you wish to quote a result that was not in your 6040 text, then you need to carefully state and prove that result.

Exam Problems:

1. Let $X_1, X_2, \ldots$ be independent random variables, and suppose that $p_i := \mathrm{P}\{X_i = 0\}$ satisfies $0 < p_i < 1$ for every $i = 1, 2, \ldots$. For every $n \geqslant 1$ let $N_n := \sum_{i=1}^n \mathbf{1}_{\{X_i = 0\}}$ denote the number of times the finite random sequence $X_1, \ldots, X_n$ enters zero. Prove that
$$\frac{N_n}{\sum_{i=1}^n p_i} \to 1 \quad \text{in probability as } n \to \infty.$$

2. Let X_n be i.i.d. random variables with mean μ and variance $\sigma^2 < \infty$. Prove (without using the law of the iterated logarithm) that we have almost surely
$$\varlimsup_{n\to\infty} \frac{S_n - n\mu}{\sigma\sqrt{n}} = \infty \quad \text{and} \quad \varliminf_{n\to\infty} \frac{S_n - n\mu}{\sigma\sqrt{n}} = -\infty.$$

3. Suppose X and Y are two independent, real-valued random variables, and suppose that $\mathrm{P}\{X \in A\} = 0$ for all Borel sets A of zero Lebesgue measure. Prove, carefully, that there exists a measurable function $f : \mathbb{R} \to \mathbb{R}$ such that
$$\mathrm{P}\{X + Y \in B\} = \int_B f(x)\,\mathrm{d}x \qquad \text{for all measurable sets } B \subseteq \mathbb{R}.$$

4. Let $X_1, X_2, \ldots$ be a collection of i.i.d. random variables, taking values ± 1 with probability $1/2$ each. Define $S_n := X_1 + \cdots + X_n$ for all $n \geqslant 1$, $T_0 := 0$, and then iteratively define
$$T_{k+1} := \inf\{n > T_k :\ S_n = 0\} \qquad [\inf \varnothing := \infty] \quad \text{for all } k \geqslant 0.$$
That is, T_k denotes the kth return time to zero. Prove that $\lim_{k\to\infty}(T_k/k) = \infty$ a.s.

5. Suppose X is a nonnegative random variable that satisfies the following: There exist two real numbers $a, b > 0$ such that $a^n \leqslant \mathrm{E}[X^n] \leqslant b^n$ for every integer $n \geqslant 1$. Prove that $\mathrm{P}\{a \leqslant X \leqslant b\} = 1$.

6. Let $X_1, \ldots, X_m$ be $m \geqslant 2$ i.i.d. standard normal random variables, and define the random vector
$$\boldsymbol{X} := \begin{bmatrix} X_1 \\ \vdots \\ X_m \end{bmatrix}.$$
Prove that the distribution of $\boldsymbol{X}$ is the same as the distribution of $\boldsymbol{AX}$ for every $m \times m$ orthogonal matrix $\boldsymbol{A}$ that is non random.

7. Suppose X is a discrete, non-negative random variable with mass function p. Prove that
$$\lim_{n\to\infty} (\mathrm{E}[X^n])^{1/n} = \sup\{x \in \mathbb{R} : p(x) > 0\}.$$

8. Fix $0 < \lambda < \rho$ and let X, Y, and Z be three independent Gamma distributed random variables with common scale parameter 1 and shape parameters λ, $\rho - \lambda$, and ρ, respectively. (A Gamma random variable with scale parameter 1 and shape parameter λ has pdf $\frac{1}{\Gamma(\lambda)} x^{\lambda-1} e^{-x} \mathbf{1}_{\{x>0\}}$.) Let
$$U = \frac{ZX}{X+Y}, \quad V = \frac{ZY}{X+Y}, \quad \text{and} \quad W = X + Y.$$
Prove that random vector (U, V, W) has the same distribution as (X, Y, Z). (It may be useful to note that $Z = U + V$.)

9. Let M_n be an L^1-bounded martingale with respect to a filtration $\mathcal{F}_n$. Prove that there exist two non-negative martingales Y_n and Z_n (in the same filtration) such that $M_n = Y_n - Z_n$. Hint: Use $Z_{n,j} = \mathrm{E}[|M_j| \,|\, \mathcal{F}_n]$ for $j \geqslant n$.

10. Let M_n be a martingale such that $\sup_n \mathrm{E}[M_n^2] < \infty$. Prove that there exists a random variable $M_\infty \in L^2$ such that $M_n \to M_\infty$ in L^2. Hint: $M_n = M_0 + \sum_{j=1}^n (M_j - M_{j-1})$.