Preliminary Examination, Numerical Analysis, August 2016

Instructions: This exam is closed books and notes. The time allowed is three hours and you need to work on any three out of questions 1-4 and any two out of questions 5-7. All questions have equal weights and the passing score will be determined after all the exams are graded. Indicate clearly the work that you wish to be graded.

Note: In problems 5-7, the notations $k = \Delta t$ and $h = \Delta x$ are used.

1. Singular Value Decomposition (SVD):

a) Prove the following statement:

Singular Value Decomposition: Any matrix $A \in \mathbb{C}^{m\times n}$ can be factored as $A = U\Sigma V^*$, where $U \in \mathbb{C}^{m\times m}$ and $V \in \mathbb{C}^{n\times n}$ are unitary and $\Sigma \in \mathbb{R}^{m\times n}$ is a rectangular matrix whose only nonzero entries are non-negative entries on its diagonal.

b) Use the SVD to prove that any matrix in $\mathbb{C}^{n\times n}$ is the limit of a sequence of matrices of full rank.

2. Linear Least Squares:

The Linear Least Squares problem for an $m \times n$ real matrix A and $b \in \mathbb{R}^m$ is the problem:

$$\text{Find} \quad \mathrm{x} \in \mathbb{R}^{\mathrm{n}} \text{ such that } \|\mathrm{Ax} - \mathrm{b}\|_2 \text{ is minimized.}$$

a) Suppose that you have data $\{(t_j, y_j)\}$, $j = 1, 2, \ldots, m$ that you wish to approximate by an expansion

$$p(t) \;=\; \sum_{k=1}^{n} x_k \phi_k(t).$$

Here, the functions $\phi_k(t)$ are given functions. Which norm on the difference between the approximation function p and the data gives rise to a linear least squares problem for the unknown expansion coefficients x_k? What is the matrix A in this case, and what is the vector b?

b) Suppose that A is a real $m \times n$ matrix of full rank and let $b \in \mathbb{R}^m$. What are the 'normal equations' for the Least Squares problem? How can they be used to solve the Least Squares problem? What is the QR factorization of A and how can it be used to solve the Least Squares problem? Compare and contrast these approaches for numerically solving the Least Square problem.

3. Sensitivity:

Consider a 6×6 symmetric positive definite matrix A with singular values $\sigma_1 = 1000$, $\sigma_2 = 500$, $\sigma_3 = 300$, $\sigma_4 = 20$, $\sigma_5 = 1$, $\sigma_6 = 0.01$.

a) Suppose you use a Cholesky factorization package on a computer with a machine epsilon 10^{-14} to solve the system $Ax = b$ for some nonzero vector b. How many digits of accuracy do you expect in the computed solution? Justify your answer in terms of condition number and stability. You may assume that the entries of A and b are exactly represented in the computer's floating-point system.

b) Suppose that instead you use an iterative method to find an approximate solution to $Ax = b$ and you stop iterating and accept iterate $x^{(k)}$ when the residual $r^{(k)} = Ax^{(k)} - b$ has 2-norm less than 10^{-9}. Give an estimate of the maximum size of the relative *error* in the final iterate? Justify your answer.

4. Interpolation and Integration:

a) Consider equally spaced points $x_j = a + jh$, $j = 0, \ldots, n$ on the interval $[a, b]$, where $nh = b - a$. Let $f(x)$ be a smooth function defined on $[a, b]$. Show that there is a unique polynomial $p(x)$ of degree n which interpolates f at all of the points x_j. Derive the formula for the interpolation error at an arbitrary point x in the interval $[a, b]$:

$$f(x) - p(x) \equiv E(x) = \frac{1}{(n+1)!}(x - x_0)(x - x_1) \cdots (x - x_n) f^{n+1}(\eta).$$

for some $\eta \in [a, b]$.

b) Let $I_n(f)$ denote the result of using the composite Trapezoidal rule to approximate $I(f) \equiv \int_a^b f(x)dx$ using n equally sized subintervals of length $h = (b - a)/n$. It can be shown that the integration error $E_n(f) \equiv I(f) - I_n(f)$ satisfies

$$E_n(f) = d_2 h^2 + d_4 h^4 + d_6 h^6 + \ldots$$

where $d_2, d_4, d_6, \ldots$ are known numbers that depend only on the values of f and its derivatives at a and b. Suppose you have a black-box program that, given f, a, b, and n, calculates $I_n(f)$. Show how to use this program to obtain an $O(h^4)$ approximation and an $O(h^6)$ approximation to $I(f)$.

5. Elliptic Problems:

For the one dimensional Poisson problem for $v(x)$

$$-v''(x) + \alpha v(x) = f(x),$$

where $\alpha \geq 0$ is constant, along with Dirichlet boundary conditions in the interval [0,1], consider the scheme

$$\Delta_h U_j \equiv \frac{1}{h^2}\Big(- U_{j-1} + 2\ U_j - U_{j+1}\Big) = f_j$$

for $j = 1, 2, \ldots, N-1$ where $Nh = 1$, $f_j \equiv f(jh)$, and $U_0 = U_N = 0$. The approximate solution satisfies a linear system $AU = b$, where $U = (U_1, U_2, ..., U_{N-1})^T$ and $b = h^2(f_1, f_2, ..., f_{N-1})^T$.

a) State and prove the maximum principle for any grid function $V = \{V_j\}$ with values for $j = 0, 1, ...N$, that satisfies $\Delta_h V_j \geq 0$.

b) Derive the matrix A and show that it is symmetric and positive definite.

c) Use the maximum principle to show that the global error $e_j = v(x_j) - U_j$ satisfies $\|e\|_\infty = O(h^2)$ as the space step $h \to 0$.

6. Numerical Methods for ODEs:

Consider the Linear Multistep Method

$$y_{n+2} - \frac{4}{3}y_{n+1} + \frac{1}{3}y_n = \frac{2}{3}kf_{n+2}$$

for solving an initial value problem $y' = f(y, x)$, $y(0) = \eta$. You may assume that f is Lipschitz continuous with respect to y uniformly for all x.

a) Analyze the consistency, stability, accuracy, and convergence properties of this method.

b) Sketch a graph of the solution to the following initial value problem.

$$y' = -10^8[y - \cos(x)] - \sin(x), \quad y(0) = 2.$$

Would it be more reasonable to use this method or the forward Euler method for this problem? What would you consider in choosing a timestep k for each of the methods? Justify your answer.

7. Heat Equation Stability:

a) Consider the initial value problem for the constant-coefficient diffusion equation (with $\beta > 0$)

$$v_t = \beta v_{xx}, \ t > 0$$

with initial data $v(x, 0) = f(x)$. A scheme for this problem is:

$$\frac{u_j^{n+1} - u_j^n}{k} = \frac{\beta}{h^2} \left\{ u_{j-1}^{n+1} - 2u_j^{n+1} + u_{j+1}^{n+1} \right\}.$$

Analyze the 2-norm stability of this scheme. For which values of $k > 0$ and $h > 0$ is the scheme stable? (Note that there are no boundary conditions here.)

b) Consider the variable coefficient diffusion equation

$$v_t = (\beta(x) v_x)_x, \qquad 0 < x < 1, \ t > 0$$

with Dirichlet boundary conditions

$$v(0,t) \ = \ 0, \qquad v(1,t) \ = \ 0$$

and initial data $v(x, 0) = f(x)$. Assume that $\beta(x) \geq \beta_0 > 0$, and that $\beta(x)$ is smooth. Let $\beta_{j+1/2} = \beta(x_{j+1/2})$. A scheme for this problem is:

$$\frac{u_j^{n+1} - u_j^n}{k} = \frac{1}{h^2} \ \left\{ \beta_{j-1/2} u_{j-1}^{n+1} - (\beta_{j-1/2} + \beta_{j+1/2}) u_j^{n+1} + \beta_{j+1/2} u_{j+1}^{n+1} \right\}.$$

Analyze the 2-norm stability of this scheme for solving this initial boundary value problem. DO NOT NEGLECT THE FACT THAT THERE ARE BOUNDARY CONDITIONS!

Fact 1: A real symmetric $n \times n$ matrix A can be diagonalized by an orthogonal similarity transformation, and A's eigenvalues are real.

Fact 2: The $(N-1) \times (N-1)$ matrix M defined by

```
[ -2  1  0  0  0  .  .  .  0  0  0  0 ]
[  1 -2  1  0  0  .  .  .  0  0  0  0 ]
[  0  1 -2  1  0  .  .  .  0  0  0  0 ]
[  0  0  1 -2  1  .  .  .  0  0  0  0 ]
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[  0  0  0  0  0  .  .  .  1 -2  1  0 ]
[  0  0  0  0  0  .  .  .  0  1 -2  1 ]
[  0  0  0  0  0  .  .  .  0  0  1 -2 ]
```

has eigenvalues $\mu_l = -4\sin^2(\frac{\pi l}{2N})$, $l = 1, 2, \ldots, N-1$.

Fact 3: The $(N+1) \times (N+1)$ matrix:

```
[ -1  1  0  0  0  .  .  .  0  0  0  0 ]
[  1 -2  1  0  0  .  .  .  0  0  0  0 ]
[  0  1 -2  1  0  .  .  .  0  0  0  0 ]
[  0  0  1 -2  1  .  .  .  0  0  0  0 ]
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[  0  0  0  0  0  .  .  .  1 -2  1  0 ]
[  0  0  0  0  0  .  .  .  0  1 -2  1 ]
[  0  0  0  0  0  .  .  .  0  0  1 -1 ]
```

has eigenvalues $\mu_l = -4\sin^2\left(\frac{\pi l}{2(N+1)}\right)$, $\quad l = 0, 1, \ldots, N$.

Fact 4: For a real $n \times n$ matrix A, the Rayleigh quotient of a vector $x \in R^n$ is the scalar

$$r(x) = \frac{x^T A x}{x^T x}.$$

The gradient of $r(x)$ is

$$\nabla r(x) = \frac{2}{x^T x}(Ax - r(x)x).$$

If x is an eigenvector of A then $r(x)$ is the corresponding eigenvalue and $\nabla r(x) = 0$.