Preliminary Exam, Numerical Analysis, January 2016

Instructions: This exam is closed book, no notes and no electronic devices are allowed. You have three hours and you need to work on any three out of questions 1-4, and any three out of questions 5-8. All questions have equal weight and a score of 75% is considered a pass. Indicate clearly the work that you wish to be graded.

Problem 1. ((**Hermitian Matrix**).
Let $A \in \mathrm{C}^{m\times m}$ be hermitian:
a) Prove that all eigenvalues of A are real.
b) Prove that if x and y are eigenvectors corresponding to distinct eigenvalues, then x and y are orthogonal.

Problem 2. (**Numerical Algorithm to Find Eigenvectors/Eigenvalues**).
State the Inverse Iteration Algorithm to find eigenvectors/eigenvalues for general real symmetric matrix $A \in \mathrm{R}^{m\times m}$.

Problem 3. (**Singular Value Decomposition**).
a) Consider $A \in \mathrm{C}^{m\times n}$. Define what we mean by the singular value decomposition of A.
b) Show that the rank of A is r, the number of nonzero singular values.

Problem 4. (**CG Method**).
a) Formulate the Conjugate Gradient (CG) algorithm for the solution of the linear system $Ax = b$, where $A \in \mathrm{R}^{n\times n}$ is symmetric positive definite matrix.
b) Show that for any initial guess $x_0 \in \mathrm{R}^n$, Conjugate Gradient algorithm converges to the solution x of the linear system $Ax = b$ in at most n steps.

Problem 5. (**Interpolation**).
a) State the theorem about the existence and uniqueness of interpolating polynomial. Give a proof. (You can consider any proof of your choice).
b) Let $f(x) = x^3 + 2x^2 + x + 1$. Find the polynomial of degree 4 that interpolates

the values of f at $x = -2, -1, 0, 1, 2$.

Problem 6. (**Linear Multistep Methods**).
Give the definition of linear multistep method (give formula).
a) State necessary and sufficient condition for linear multistep method to be consistent.
b) State root condition for linear multistep method.
c) Construct an example of a consistent but not stable linear multistep method. Justify your answer.

Problem 7. (**Region of Absolute Stability and Example of Linear One-Step Method**).
1. Define a region of absolute stability for Linear Multistep Method.
2. Find the region of absolute stability for the Forward Euler Method.

Problem 8. (**Upwind Scheme**).
Consider the advection equation

$$u_t + 5u_x = 0, \quad x_L < x < x_R, \quad 0 < t \leq T,$$

where $u(x,0) = g(x)$, and $u(x_L, t) = u_L(t)$ for $t > 0$
a) Write the Upwind Scheme for this problem.
b) What is the local truncation error of the Upwind Scheme?
c) What is the stencil of the scheme? What is the CFL condition for this method?
d) Investigate the stability of the method using Von Neumann Stability Analysis.