DEPARTMENT OF MATHEMATICS

University of Utah

Ph.D. PRELIMINARY EXAMINATION IN GEOMETRY/TOPOLOGY

January 2007

Instructions: Do four (4) problems from section A and four (4) problems from section B. Be sure to provide all relevant definitions and statements of theorems cited. Make sure you indicate which solutions are to be graded, otherwise the first four answered will be scored. For a pass, three problems from each group have to be solved entirely.

A. Answer four of the following questions. Each question is worth ten points. All manifolds are smooth and they are submanifolds of some Euclidean space.

1. Let S^n be the unit sphere in $\mathbb{R}^{n+1}$ and $h : \mathbb{R}^{n+1} \to \mathbb{R}$ the projection to the last coordinate. Prove that the restriction of h to S^n is a Morse function and find all critical points and their indices.

2. Find a perturbation of the identity map $\mathbb{R}P^3 \to \mathbb{R}P^3$ which is a Lefschetz map and compute its Lefschetz number.

3. Give a proof that $\mathbb{R}P^2$ is nonorientable.

4. Compute de Rham cohomology of the 2-torus. You may use the Poincaré lemma and the Mayer-Vietoris sequence. For example, you may want to compute $H^*(S^1)$ first.

5. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Show that for every $\epsilon > 0$ there is $v \in \mathbb{R}^n$ with $|v| < \epsilon$ such that the function $g_v : \mathbb{R}^n \to \mathbb{R}$ given by

$$g_v(x) = f(x) + v \cdot x$$

is Morse.

6. The set $SL_2(\mathbb{R})$ of 2×2 matrices with determinant 1 can be viewed as a subset of $\mathbb{R}^4$ by choosing an ordering of the matrix entries. Show that this set is a submanifold of $\mathbb{R}^4$ and compute the tangent space at the identity matrix.

7. Consider the vector fields in $\mathbb{R}^3$:

$$X = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \quad \text{and} \quad Y = \frac{\partial}{\partial x} - \frac{\partial}{\partial z}$$

Show that there is no nonempty surface $S \subset \mathbb{R}^3$ that is tangent to both vector fields at each of its points.

8. Show that for any two points $x, y \in \mathbb{R}^n$ there is a compactly supported isotopy ϕ_t (i.e. it is identity outside a compact set) such that $\phi_0 = id$ and $\phi_1(x) = y$.

B. Answer four of the following questions. Each question is worth ten points.

9. Define a Δ-complex structure on the Klein bottle and compute its homology with $\mathbb{Z}$ and $\mathbb{Z}/2$ coefficients.

10. Give an example of an irregular (i.e. not normal) covering space (with a proof).

11. Let $f : M \to N$ be a map of degree 1 between two smooth, closed, connected, oriented n-manifolds. Prove that $f_\# : \pi_1(M) \to \pi_1(N)$ is surjective.

12. Let M be a closed connected 5-manifold such that $\pi_1(M) \cong \mathbb{Z}/7$. If $H_2(M;\mathbb{Z}) \cong \mathbb{Z}$, compute all other homology and cohomology groups of M with integral coefficients.

13. Let F_n be the free group of rank n and let $G \subset F_n$ be a subgroup of index m. Prove that G is a free group and compute its rank.

14. Define carefully a CW structure on $\mathbb{R}P^n$ (you don't have to prove it here), and use it to compute $H_*(\mathbb{R}P^n;\mathbb{Q})$ and $H_*(\mathbb{R}P^n;\mathbb{Z}/2)$.

15. Prove that the map $h : S^3 \to \mathbb{C}P^1$ given by $h(x,y) = [x : y]$ is a fiber bundle. Here, S^3 is the unit sphere $|x|^2 + |y|^2 = 1$ in $\mathbb{C}^2$.

16. Prove that $\mathbb{R}P^3$ is not homotopy equivalent to $\mathbb{R}P^2 \vee S^3$.

DEPARTMENT OF MATHEMATICS

University of Utah

Ph.D. PRELIMINARY EXAMINATION IN GEOMETRY/TOPOLOGY

August 21, 2006, 9:00-12:00

Instructions: Do four (4) problems from section A and four (4) problems from section B. Be sure to provide all relevant definitions and statements of theorems cited. Make sure you indicate which solutions are to be graded, otherwise the first four answered will be scored. For a pass, three problems from each group have to be solved entirely.

A. Answer four of the following questions. Each question is worth ten points. All manifolds are smooth and they are submanifolds of some Euclidean space.

1. Show that the set M_1 of real 2×2 matrices of rank 1 is a 3-dimensional submanifold of the space $M(2,2) \cong \mathbb{R}^4$ of all real 2×2 matrices.

2. For which values of $a > 0$ does the hyperboloid
$$x^2 + y^2 - z^2 = 1$$
intersect the sphere $x^2 + y^2 + z^2 = a$ transversally (in $\mathbb{R}^3$)?

3. Let $X, Y \subset \mathbb{R}^3$ be two 1-dimensional submanifolds. Show that there is $v \in \mathbb{R}^3$ such that X is disjoint from $Y + v := \{y + v | y \in Y\}$.

4. Define the Lefschetz index of an isolated fixed point of a smooth map $f : M \to M$. Compute the Lefschetz index at 0 of the map $f : \mathbb{C} \to \mathbb{C}$ given by $f(z) = z + z^m$, $m > 0$.

5. Compute the Gaussian curvature of the hyperboloid
$$x^2 + y^2 - z^2 = 1$$
at the point $(1, 0, 0)$.

6. Regard the real projective plane $\mathbb{R}P^2$ as the space of triples $(x, y, z) \in \mathbb{R}^3 \setminus \{0\}$ modulo the relation $(x, y, z) \sim (tx, ty, tz)$ for $t \in \mathbb{R} \setminus \{0\}$. Define
$$f : \mathbb{R}P^2 \to \mathbb{R}$$
by
$$f(x, y, z) = \frac{x^2 + 2y^2}{x^2 + y^2 + z^2}$$
Compute all critical points of f. Show that f is a Morse function and compute the Morse index of each critical point.

7. Give an explicit example of a closed 1-form on $\mathbb{R}^2 - \{0\}$ which is not exact, and prove both properties.

8. Verify the formula

$$d\omega(X,Y) = X(\omega(Y)) - Y(\omega(X)) - \omega([X,Y])$$

for any 1-form ω on $\mathbb{R}^n$ and any two vector fields X, Y on $\mathbb{R}^n$. This is using Spivak's normalization conventions ($dx \wedge dy(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}) = 1$); using Guillemin-Pollack's one should multiply the left-hand side by 2.

B. Answer four of the following questions. Each question is worth ten points.

9. Let X be the quotient space of S^2 obtained by identifying the north and south poles to a single point. Carefully define a cell complex structure on X (including a description of attaching maps) and use this to compute $\pi_1(X)$.

10. For a connected CW complex X, call a connected covering space $\tilde{X} \to X$ *abelian* if it is normal and has abelian deck transformation group. Show that X has an abelian covering space that is a covering space of every other abelian covering space of X, and that such 'universal' abelian covering space is unique up to isomorphism. Describe this covering space explicitly for $X = S^1 \vee S^1$. Carefully state theorems you are using.

11. Define a Δ-complex structure on the Klein bottle K and use it to compute $H_*(K;\mathbb{Z})$, $H_*(K;\mathbb{Z}_2)$, and $H_*(K;\mathbb{Q})$.

12. Let X be the quotient space of the 2-sphere S^2 under the identifications $x \sim -x$ for x in the equator S^1. Compute the homology groups $H_i(X)$ using any method you like.

13. Apply the Lefschetz fixed point theorem to show that every map $f : \mathbb{C}P^n \to \mathbb{C}P^n$ has a fixed point when n is even (state the Lefschetz fixed point theorem and any fact about the ring structure of $H^*(\mathbb{C}P^n)$ you are using). Construct a fixed point free map $f : \mathbb{C}P^n \to \mathbb{C}P^n$ when n is odd.

14. Show that a p-sheeted covering map $M \to N$ between closed connected oriented smooth manifolds has degree $\pm p$.

15. Show that if a closed orientable manifold M of dimension $2k$ has $H_{k-1}(M;\mathbb{Z})$ torsionfree, then $H_k(M;\mathbb{Z})$ is also torsionfree.

16. Show that if the closed orientable surface M_g of genus g retracts onto a graph $X \subset M_g$, then $H_1(X)$ has rank at most g. You may use the following algebraic fact: a nonsingular skew-symmetric bilinear pairing over the rationals $\mathbb{Q}$, of the form $\mathbb{Q}^n \times \mathbb{Q}^n \to \mathbb{Q}$, cannot be identically 0 when restricted to $V \times V$ for any $\mathbb{Q}$-linear subspace $V \subset \mathbb{Q}^n$ of dimension $> n/2$.

DEPARTMENT OF MATHEMATICS
University of Utah
Ph.D. PRELIMINARY EXAMINATION IN GEOMETRY/TOPOLOGY
January 4, 2006, 9:00-12:00, LCB 215

Instructions: Do four (4) problems from section A and four (4) problems from section B. Be sure to provide all relevant definitions and statements of theorems cited. Make sure you indicate which solutions are to be graded, otherwise the first four answered will be scored. For a pass, three problems from each group have to be solved entirely.

A. Answer four of the following questions. Each question is worth ten points. All manifolds are smooth and they are submanifolds of some Euclidean space.

1. Let X be a manifold and $f : X \to \mathbb{R}^k$ a continuous function. Show that for every $\epsilon > 0$ there is a smooth function $g : X \to \mathbb{R}^k$ such that
$$||g(x) - f(x)|| < \epsilon$$
for every $x \in X$.

2. Let $\mathcal{M}_{n\times n}$ be the set of all real $n\times n$ matrices. This set is naturally a manifold, since it can be identified with $\mathbb{R}^{n^2}$ by choosing an ordering of entries. Let $O(n) \subset \mathcal{M}_{n\times n}$ be the set of orthogonal matrices, i.e. matrices A with $AA^\top = I$. Prove that $O(n)$ is a submanifold of $\mathcal{M}_{n\times n}$. What is the dimension of $O(n)$?

3. (a) Give a careful definition of the tangent bundle of a manifold X.
   (b) Show that the total space of the tangent bundle of S^2 is not diffeomorphic to $S^2 \times \mathbb{R}^2$.

4. (a) State the Lefschetz fixed point theorem (for Lefschetz maps).
   (b) Compute the Lefschetz number of the identity map $id : \mathbb{C}P^2 \to \mathbb{C}P^2$ by first perturbing it to a Lefschetz map and then applying the Lefschetz fixed point theorem.

5. (a) State the Frobenius integrability theorem.

(b) Show that the plane field in $\mathbb{R}^3$ spanned by the vector fields

$$X = \frac{\partial}{\partial x} + z\frac{\partial}{\partial y} \qquad \text{and} \qquad Y = \frac{\partial}{\partial z}$$

is not integrable on any nonempty open subset of $\mathbb{R}^3$.

6. Let ω be a compactly supported smooth n-form on $\mathbb{R}^n$. Prove that the following two statements are equivalent.

(a) There exists a compactly supported smooth $(n-1)$-form η such that $\omega = d\eta$.

(b) $\int_{\mathbb{R}^n} \omega = 0$.

Note: You are allowed to use the fact that $H^n(S^n) = \mathbb{R}$, but you are **not** allowed to quote the fact that $H^n_c(\mathbb{R}^n) = \mathbb{R}$.

7. (a) Let $a : S^n \to S^n$ be the antipodal map. Suppose that ω is a smooth form on S^n such that $a^*\omega = \omega$. Prove that if ω is exact, then there is a smooth form η with $\omega = d\eta$ and $a^*\eta = \eta$.

(b) Use (a) and the fact that $H^k_{DR}(S^n) = 0$ for $0 < k < n$ to deduce that $H^k_{DR}(\mathbb{R}P^n) = 0$ for $0 < k < n$.

8. (a) State the tubular neighborhood theorem for a compact submanifold $X \subset \mathbb{R}^n$.

(b) Give an outline of the proof with details of the step that shows that the constructed map is a diffeomorphism.

$$\int_{\mathbb{R}^n} d\eta = \int_{\partial\mathbb{R}^n} \eta$$

B. Answer four of the following questions. Each question is worth ten points.

9. Two covering spaces $p_0 : X_0 \to X$ and $p_1 : X_1 \to X$ of a space X are isomorphic if there is a homeomorphism $h : X_0 \to X_1$ such that $p_0 = p_1 \circ h$. In this question all spaces are path-connected, locally path-connected and semilocally path-connected.

   (a) Prove the following piece of the classification theorem for covering spaces: Let $x \in X$, $x_0 \in X_0$ and $x_1 \in X_1$ be basepoints such that $p_0(x_0) = p_1(x_1) = x$. The covers are isomorphic if and only if $(p_0)_*(\pi_1(X_0, x_0))$ and $(p_1)_*(\pi_1(X_1, x_1))$ are conjugate subgroups of $\pi_1(X, x)$.

   (b) Show that there are non-isomorphic covers of a genus two surface of every degree.

10. Let M be an open 3-manifold such that every compact set of M is contained in a solid torus.

   (a) Show that $\pi_1(M)$ is abelian.

   (b) Assume that $\pi_1(M)$ is finitely generated. Show that $\pi_1(M)$ is trivial or $\mathbb{Z}$.

11. Let c be a separating simple closed curve on a compact surface S without boundary. Show that S doesn't retract onto c.

12. Construct a Δ-complex structure for a genus two surface. Use the Δ-complex to calculate the cohomology and cup product structure for the surface.

13. Show that $S^2 \vee S^4$ and $\mathbb{C}P^2$ are not homotopy equivalent.

14. Let X_d be the CW-complex obtained by attaching a disk with a degree d map to the circle $S^1 \times \{x_0\}$ on the torus $S^1 \times S^1$ (and $x_0 \in S^1$ is a basepoint). Calculate the homology and cohomology of X_d with $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{Z}_p$ coefficients where p is prime.

15. Let M_0 and M_1 be orientable closed n-manifolds and $f : M_0 \to M_1$ a map such that $f^* : H^n(M_1; \mathbb{R}) \to H^n(M_0; \mathbb{R})$ is an isomorphism.

(a) Assume that $H^i(M_0; \mathbb{R}) \cong H^i(M_1; \mathbb{R})$ for all i. Show that f^* is an isomorphism on cohomology with $\mathbb{R}$ coefficients in all dimensions.

(b) Assume that there exists a map $g : M_1 \to M_0$ such that $g^* : H^n(M_0; \mathbb{R}) \to H^n(M_1; \mathbb{R})$ is an isomorphism. Show that f^* and g^* are isomorphisms on cohomology with $\mathbb{R}$ coefficients in all dimensions.

16. Define the higher homotopy groups, π_i, for $i \geq 2$ and prove that they are abelian.

DEPARTMENT OF MATHEMATICS
University of Utah
Ph.D. PRELIMINARY EXAMINATION IN GEOMETRY/TOPOLOGY
August 17, 2005, 9:00-12:00, LCB 219

Instructions: Do four (4) problems from section A and four (4) problems from section B. Be sure to provide all relevant definitions and statements of theorems cited. Make sure you indicate which solutions are to be graded, otherwise the first four answered will be scored. For a pass, three problems from each group have to be solved entirely.

A. Answer four of the following questions. Each question is worth ten points. All manifolds are smooth and they are submanifolds of some Euclidean space.

1. In this problem we identify $\mathbb{C}$ with $\mathbb{R}^2$ in the usual manner via

$$z \mapsto (\Re z, \Im z)$$

Consider the following smooth submanifolds of $\mathbb{C}^2 - \{(0,0),(0,0.5)\}$:

$$\begin{aligned} M_1 =& \{(z,w) \in \mathbb{C}^2 | z^2 + w^2 = 1\} \\ M_2 =& \{(z,w) \in \mathbb{C}^2 | z^2 - w^2 + w = 1\} \\ M_3 =& \{(x_1,y_1,x_2,y_2) \in \mathbb{R}^4 | 1 - x_1 + x_2^2 + y_1^2 + y_2^2 = 0\} \end{aligned}$$

Which of the pairs M_i, M_j $(i,j \in \{1,2,3\})$ are transverse at $p = (1,0) \in \mathbb{C}^2$?

2. Let $V, W : \mathbb{R}^4 \to \mathbb{R}^4$ be vector fields on $\mathbb{R}^4$ defined by

$$V(x,y,z,w) = (y, -x, w, -z)$$

and

$$W(x,y,z,w) = (w, z, -y, -x)$$

Is there a nonempty surface $\Sigma \subset \mathbb{R}^4$ such that for every $p \in \Sigma$ we have $V(p), W(p) \in T_p\Sigma$? Find such a surface or prove that it does not exist.

3. Let M be a nonempty compact manifold with empty boundary. Prove that there is no compact manifold $W \subset M \times M$ whose boundary ∂W is the diagonal $\Delta = \{(x,x) \in M \times M\}$.

4. Is there a smooth embedding of $\mathbb{R}P^2$ into an orientable 3-manifold? Provide a proof or an example.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be the function given by $f(x,y) = (x^2 - y, x + y^2, sin(x))$. Let ω be the 2-form on $\mathbb{R}^3$ given by $\omega(u,v,w) = u dv \wedge dw$. Compute the pull-back $f^*\omega$.

6. Prove that for any two disjoint closed subsets $A, B \subset \mathbb{R}^n$ there is a smooth function $f : \mathbb{R}^n \to \mathbb{R}$ such that $f = 0$ on A and $f = 1$ on B.

7. Show that the total space of the tangent bundle $T(S^2)$ of the 2-sphere is not diffeomorphic to $S^2 \times \mathbb{R}^2$. Hint: One strategy is to find a 2-sphere in $T(S^2)$ whose self-intersection number is nonzero and argue that there are no such 2-spheres in $S^2 \times \mathbb{R}^2$.

8. Compute the curvature of the hyperboloid $x^2 + y^2 - z^2 = 1$ at the point $(1, 0, 0)$.

B. Answer four of the following questions. Each question is worth ten points.

9. Suppose $p : \tilde{X} \longrightarrow X$ is a covering map. Prove that p is null homotopic if and only if $\tilde{X}$ is contractible.

10. Let $T^2 = S^1 \times S^1$ be the standard 2-torus and $p, q \in T^2$ be two distinct points. Let $X = T^2/\sim$ be the identification space where the two points are identified $p \sim q$. Find $\pi_1(X)$.

11. Carefully construct a Δ-complex structure on $\mathbb{R}P^2$ and use it to compute the homology groups of $\mathbb{R}P^2$ with $\mathbb{Z}$ and $\mathbb{Z}_2$ coefficients.

12. Let $f : S^n \longrightarrow S^n$ be a continuous map from the n-sphere to itself and assume that n is even. Show that there is a point $x \in S^n$ such that $f(x) \in \{x, -x\}$. Does the same assertion hold if n is odd? If it does, prove it, and if it does not, find a counterexample (for every odd n).

13. (a) Describe the (simplest) cell structure of $\mathbb{R}P^n$ (give the attaching maps of all the cells - you don't have give a proof here).
    (b) Write down the cellular chain complex associated with this cell decomposition.
    (c) Use this chain complex to compute the homology and cohomology groups of $\mathbb{R}P^n$ with coefficients in $\mathbb{Z}$ and $\mathbb{Z}_2$.

14. Prove that $\mathbb{R}P^3$ and $\mathbb{R}P^2 \vee S^3$ are not homotopy equivalent. (Hint: Cup products.)

15. Let Σ be a simply connected compact 3-manifold. Prove that $H_k(\Sigma) \cong H_k(S^3)$ for any integer k.

16. Let $p : S^3 \longrightarrow \mathbb{C}P^1 = S^2$ be the map that sends $(z, w) \in S^3 \subset \mathbb{C}^2$ to its equivalence class $[z : w] \in \mathbb{C}P^1$.
    (a) Prove that p is a fiber bundle by explicitly writing down local trivializations.
    (b) Write down the associated long exact sequence of homotopy groups.
    (c) Compute $\pi_3(S^2)$. Justify all auxiliary results you are using.

DEPARTMENT OF MATHEMATICS
University of Utah
Ph.D. PRELIMINARY EXAMINATION IN GEOMETRY/TOPOLOGY
January, 2005, 9:00-12:00, JWB 333

Instructions: Do five (5) problems from section A and five (5) problems from section B. Be sure to provide all relevant definitions and statements of theorems cited. Make sure you indicate which solutions are to be graded, otherwise the first five answered will be scored. For a pass, three problems from each group have to be solved entirely.

A. Answer five of the following questions. Each question is worth ten points. All manifolds are smooth and they are submanifolds of some Euclidean space.

1. Let
$$X = \{(x, y) \in \mathbb{R}^3 \times \mathbb{R}^3 | x \cdot x = y \cdot y = 1, x \cdot y = 0\}$$
Show that X is a submanifold of $\mathbb{R}^3 \times \mathbb{R}^3$.

2. Prove that the tangent bundle TM of a smooth manifold M, viewed as a smooth manifold, is orientable.

3. Let $f : M \to N$ be a map between two closed connected oriented manifolds. If f has degree 1, show that $f_* : \pi_1(M) \to \pi_1(N)$ is an epimorphism.

4. Let M be a manifold and Δ a smooth k-plane field on M. Suppose that X and Y are smooth vector fields on M with values in Δ. Show that if for some $p \in M$ we have $X_p = 0$ then $[X, Y]_p \in \Delta$.

5. Let
$$X = y\frac{\partial}{\partial x} - (x+1)\frac{\partial}{\partial y}$$
and
$$Y = y\frac{\partial}{\partial x} - (x-1)\frac{\partial}{\partial y}$$
be two vector fields in $\mathbb{R}^2$. Compute $[X, Y]$.

6. Let X be the standard "middle thirds" Cantor set in $\mathbb{R}$, viewed as a subset of the plane $\mathbb{R}^2$. Assume that $f : X \to \mathbb{R}$ is a function with the property that for every $x \in X$ there is an open set $U_x \ni x$ in $\mathbb{R}^2$ and a smooth function $f_x : U_x \to \mathbb{R}$ such that $f_x|U_x \cap X = f|U_x \cap X$. Show that there is an open set U in $\mathbb{R}^2$, $U \supset X$, and a smooth function $g : U \to \mathbb{R}$ such that $g|X = f$.

7. Prove or give a counterexample (with proofs): If M is a closed submanifold of $\mathbb{R}^n$ then its tangent bundle is trivial if and only if its normal bundle is trivial.

8. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Show that for almost every $a \in \mathbb{R}^n$ the function

$$f_a : \mathbb{R}^n \to \mathbb{R}$$

defined by

$$f_a(x) = f(x) + x \cdot a$$

is Morse.

9. Let

$$\omega = (x^2 + y^2 + z^2)^\alpha (x dy \wedge dz + y dz \wedge dx + z dx \wedge dy)$$

be a 2-form in $\mathbb{R}^3 \setminus \{0\}$.

(a) Find $\alpha \in \mathbb{R}$ such that ω is closed.

(b) Show that ω is not exact.

10. Let X and Y be two closed oriented manifolds. Prove that Euler characteristic is multiplicative, i.e. that

$$\chi(X \times Y) = \chi(X) \times \chi(Y)$$

You may not appeal to the Künneth formula.

B. Answer five of the following questions. Each question is worth ten points.

11. Find a non-regular cover of $S^1 \vee S^1$, the wedge of 2 circles. You should both explicitly describe the cover and the corresponding subgroup of $\pi_1(S^1 \vee S^1)$.

12. Let X be the space obtained from two tori, $S^1 \times S^1$, by gluing $S^1 \times \{pt\}$ in one tours to $S^1 \times \{pt\}$ in the other torus with a degree two map. Calculate $\pi_1(X)$.

13. Let K be the Δ-complex obtained by identifying all the 2-dimensional faces of a 3-simplex. Calculate the simplicial homology of K.

14. Let
$$f : S^1 \times S^1 \longrightarrow S^1 \times S^1$$
be map of the torus to itself. Since $H_1(S^1 \times S^1) = \mathbb{Z} \oplus \mathbb{Z}$, after choosing generators for $H_1(S^1 \times S^1)$ we can represent
$$f_* : H_1(S^1 \times S^1) \longrightarrow H_1(S^1 \times S^1)$$
as a 2×2 matrix A with integer entries. If f does not have fixed points what are the possibilities for the trace and determinant of A and the degree of f? For each possible triple give an example of a fixed point free map.

15. Construct examples of CW-complexes X with each of the following properties.
    (a) For some n, $H_n(X) \neq H^n(X;\mathbb{Z})$.
    (b) For some n, $H_n(X;\mathbb{Z}_2) \neq 0$ and $H_n(X;\mathbb{Z}_3) \neq 0$ but $H_n(X;\mathbb{Z}_5) = 0$.

16. Show that $S^2 \vee S^4$ is not homotopy equivalent to $\mathbb{C}P^4$.

17. Let M be a compact manifold with boundary. Show that M does not retract onto ∂M.

18. Let M be a closed, orientable manifold M of dimension $2k$. If $H_{k-1}(M)$ is torsion free show that $H_k(M)$ is torsion free.

19. (a) Let X be a CW-complex and A a sub-complex. Assume that all cells of $X - A$ have dimension $> n$. If X retracts to A show that $\pi_n(X) = \pi_n(A)$.
    (b) Show that $\mathbb{R}P^n$ does not retract to $\mathbb{R}P^k$ if $k < n$.

20. Calculate $\pi_3(S^2)$.

DEPARTMENT OF MATHEMATICS
University of Utah
Ph.D. PRELIMINARY EXAMINATION IN GEOMETRY/TOPOLOGY
August 18, 2004, 9:00-12:00, JWB 333

Instructions: Do five (5) problems from section A and five (5) problems from section B. Be sure to provide all relevant definitions and statements of theorems cited. Make sure you indicate which solutions are to be graded, otherwise the first five answered will be scored.

A. Answer five of the following questions. Each question is worth ten points. All manifolds are smooth and they are submanifolds of some Euclidean space.

1. Let $M \subset \mathbb{R}^n$ be a compact manifold without boundary. Prove that there is $\epsilon > 0$ such that for any manifold X the following holds: any two smooth maps

$$f, g : X \to M$$

with $|f(x) - g(x)| < \epsilon$ for every $x \in X$ are homotopic. Find an explicit $\epsilon > 0$ for the case when M is the unit sphere S^{n-1}.

2. Regard the real projective plane $\mathbb{R}P^2$ as the space of triples $(x, y, z) \in \mathbb{R}^3 \setminus \{0\}$ modulo the relation $(x, y, z) \sim (tx, ty, tz)$ for $t \in \mathbb{R} \setminus \{0\}$. Define

$$f : \mathbb{R}P^2 \to \mathbb{R}$$

by

$$f(x, y, z) = \frac{x^2 + 2y^2}{x^2 + y^2 + z^2}$$

Compute all critical points of f. Show that f is a Morse function and compute the Morse index of each critical point. (Hint: Work in local coordinates, in standard charts U_0, U_1, U_2 where $U_0 = \{x \neq 0\}$ etc.)

3. Compute the Lefschetz number of the mapping

$$f_k : \mathbb{C}P^n \to \mathbb{C}P^n$$

given by

$$[x_0 : x_1 : \cdots : x_n] \mapsto [x_0^k : x_1^k : \cdots : x_n^k]$$

for $k = 1, 2, \cdots$.

4. Let p be a homogeneous polynomial in k real variables. This means that

$$p(tx_1, tx_2, \cdots, tx_k) = t^m p(x_1, x_2, \cdots, x_k)$$

for all $t, x_1, x_2, \cdots, x_k \in \mathbb{R}$. Prove that for $a \neq 0$ the set

$$\{x \in \mathbb{R}^k | p(x) = a\}$$

is a submanifold of $\mathbb{R}^k$.

5. Let X and Y be two submanifolds of $\mathbb{R}^n$. Show that for almost every $a \in \mathbb{R}^n$ the translate

$$X + a = \{x + a | x \in X\}$$

is transverse to Y.

6. Prove that the Euler characteristic of the orthogonal group $O(n)$ is 0 for $n \geq 1$.

7. Let V be the vector field in the complex plane $\mathbb{C}$ given by

$$V(z) = z^m$$

for some $m = 1, 2, \cdots$, where we use the standard identification $T_z\mathbb{C} = \mathbb{C}$. Compute the index of V at the origin.

8. Let ω be the 1-form on $\mathbb{R}^2 - \{0\}$ given by

$$\omega(x, y) = \frac{-y\, dx}{x^2 + y^2} + \frac{x\, dy}{x^2 + y^2}$$

   (a) Calculate $\int_C \omega$ for any circle C centered at the origin.
   (b) Prove that in the half-plane $x > 0$ there is a smooth function f such that $\omega = df$.
   (c) Prove that there is no smooth function f defined on $\mathbb{R}^2 - \{0\}$ such that $\omega = df$.

9. Give an example (with proof) of a nonintegrable plane field.
10. Compute de Rham cohomology of the 3-torus $S^1 \times S^1 \times S^1$.

B. Answer five of the following questions. Each question is worth ten points.

11. Let P be a discrete set of points in $\mathbb{R}^n$. Show that $\mathbb{R}^n \backslash P$ is simply connected if $n \geq 3$.
12. Find all connected covering spaces of $\mathbb{R}P^2 \vee S^1$ of degree less than three. Which of these coverings are regular?
13. Construct a Δ-complex for the Klein bottle and calculate its simplicial homology for both $\mathbb{Z}$ and $\mathbb{Z}_2$ coefficients.
14. Compute $H_i(S^n - X)$ when X is a subspace of S^n homeomorphic to $S^k \vee S^\ell$.
15. Construct a homeomorphism $f : S^1 \times S^1 \longrightarrow S^1 \times S^1$ that is not homotopic to a map without fixed points.
16. Let X be the space obtained by attaching a disk D to $S^1 \times S^1$ where the attaching map is a degree d map from ∂D to $S^1 \times \{pt\}$. Calculate the homology and cohomology groups for X with $\mathbb{Z}$ and $\mathbb{Z}_p$ coefficients where p is prime.
17. Let S_g be the closed, orientable surface of genus g. Show that there exists a map $f : S_g \longrightarrow S_h$ with

$$f_* : H_2(S_g) \longrightarrow H_2(S_h)$$

non-trivial if and only if $g \geq h$.

18. Let $X = S(S^1 \times S^1)$ be the suspension of the torus and $Y = SX$ the suspension of X. Show that X and Y aren't manifolds.
19. Let X be a connected CW-complex obtained by attaching an $n+2$-dimensional cell to an n-dimensional CW-complex A. What is the smallest possible dimension k where $\pi_k(X)$ is not isomorphic to $\pi_k(A)$?
20. (a) Let S_g be the closed, orientable surface of genus g. Calculate $\pi_2(S_g)$ for all g.
    (b) Show that $\pi_3(S_0)$ is non-trivial. (S_0 is the 2-sphere.)

DEPARTMENT OF MATHEMATICS
University of Utah
Ph.D. PRELIMINARY EXAMINATION IN GEOMETRY/TOPOLOGY
January 2004

Instructions: Do five (5) problems from section A and five (5) problems from section B. Be sure to provide all relevant definitions and statements of theorems cited. Make sure you indicate which solutions are to be graded, otherwise the first five answered will be scored.

A. Answer five of the following questions. Each question is worth ten points. All manifolds are smooth and they are submanifolds of some Euclidean space.

1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function. Show that there is $v \in \mathbb{R}^n$ such that the function $g_v : \mathbb{R}^n \to \mathbb{R}$ given by

$$g_v(x) = f(x) + v \cdot x$$

is Morse (i.e. all critical points are nondegenerate).

2. Identify the set $M(n)$ of all $n \times n$ matrices with real entries with $\mathbb{R}^{n^2}$ as usual. Show that the set

$$O(n) = \{M \in M(n) | MM^t = I\}$$

is a submanifold of $\mathbb{R}^{n^2}$ (here M^t denotes the transpose of M and I is the identity matrix).

3. Consider vector fields in $\mathbb{R}^3$:

$$X = x\frac{\partial}{\partial x} + \frac{\partial}{\partial y} \qquad Y = \frac{\partial}{\partial x} - \frac{\partial}{\partial z}$$

Show that there is no nonempty surface $S \subset \mathbb{R}^3$ that is tangent to both vector fields at each of its points.

4. Let ω be a compactly supported smooth n-form on $\mathbb{R}^n$. Show that there exists a compactly supported smooth $(n-1)$-form η with $\omega = d\eta$ if and only if

$$\int_{\mathbb{R}^n} \omega = 0$$

You are allowed to quote facts about $H^k(X)$ but not about $H^k_c(X)$.

5. Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$f(x, y) = (x^2 + y, x - y^2)$$

Explicitly compute $f^*dx, f^*dy, f^*(dx \wedge dy)$.

6. Show that the product $M \times N$ of two manifolds is orientable if and only if both M and N are orientable.

7. Using winding numbers, show that there is a complex number z such that

$$z^2 = e^{-|z|^2} + ie^{-|z|}$$

8. Let M be a manifold and $f : M \to \mathbb{R}$ a continuous function. Show that there is a smooth function $g : M \to \mathbb{R}$ such that

$$|f(x) - g(x)| < 1$$

for every $x \in M$.

9. Let $f : \Sigma \to S^1$ be a smooth map from the surface of genus 2 to the circle. Suppose that $y \in S^1$ is a regular value of f and that $f^{-1}(y)$ is a nonseparating circle in Σ (i.e. $\Sigma \setminus f^{-1}(y)$ is connected). Prove that f is not nullhomotopic.

10. Let $Y \subset \mathbb{R}^n$ be a compact manifold without boundary. Prove that there is $\epsilon > 0$ with the following property. If $f, g : X \to Y$ are smooth maps defined on a manifold X and if $|f(x) - g(x)| < \epsilon$ for every $x \in X$, then f and g are homotopic.

B. Answer five of the following questions. Each question is worth ten points.

11. Give an example of a topological space that does not have a universal cover.

12. Let M be a 3-manifold such that for every compact set $K \subset M$ there is a solid torus $W \subset M$ with $K \subset W$. Show that $\pi_1(M) = \mathbb{Z}$ or 0.

13. (a) Let M be a manifold with boundary. Show that if M is orientable then ∂M is orientable.

    (b) Let M be an n-dimensional manifold and $N \subset M$ an $n-1$ dimensional submanifold such that N has a neighborhood $U \subset M$ with U homeomorphic to $N \times (0, 1)$. Show that if M is orientable then N is orientable.

14. Let S be a closed, orientable surface and γ a simple closed curve on S. Show that S retracts onto γ if and only if γ represents a non-trivial element in homology.

15. Show that $\mathbb{R}P^3$ is not homotopy equivalent to $\mathbb{R}P^2 \vee S^3$.

16. Let M be a closed 4-manifold, S a surface with $\chi(S) \leq 0$ and $p : M \longrightarrow S$ a fiber bundle. Show that there exists a closed surface F with $\pi_n(M) = \pi_n(F)$ for $n \geq 2$.

17. Let M be a Möbius strip and D a disk. Let $f : \partial D \longrightarrow \partial M$ have degree n. Set $X_n = M \amalg D/\sim$ where $x \sim y$ if $f(x) = y$. Calculate $\pi_1(X_k)$ and construct all of its connected covering spaces.

18. (a) Let M be a closed, orientable 3-manifold such that

$$H_1(M; \mathbb{Z}) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$$

    Calculate the remaining homology and cohomology groups for M with $\mathbb{Z}$ coefficients.

    (b) Now assume that M is a closed, non-orientable 3-manifold. Show that $H_1(M; \mathbb{Z})$ is infinite.

19. Let X be a topological space obtained by adding a k-cell to $\mathbb{C}P^n$. For what values i must $H_i(X; \mathbb{Z}) = H_i(\mathbb{C}P^n; \mathbb{Z})$? What are the possiblitites for $H_i(X; \mathbb{Z})$ when the two groups are not equal?

20. Let $f : S^n \longrightarrow S^n$ be a continuous map from the n-sphere to itself and assume that n is even. Show that there is a point $x \in S^n$ such that $f(x) \in \{-x, x\}$. Does the same assertion hold if n is odd? If it does, prove it, if it does not construct a counterexample.

DEPARTMENT OF MATHEMATICS
University of Utah
Ph.D. PRELIMINARY EXAMINATION IN GEOMETRY/TOPOLOGY
August 2003

Instructions: Do five (5) problems from section A and five (5) problems from section B. Be sure to provide all relevant definitions and statements of theorems cited. Make sure you indicate which solutions are to be graded, otherwise the first five answered will be scored.

A. Answer five of the following questions. Each question is worth ten points.

1. Let X and Y be closed submanifolds of $\mathbb{R}^n$. Show that for almost all $a \in \mathbb{R}^n$ the translate $X + a = \{x + a | x \in X\}$ intersects Y transversally.

2. Let $f : X \to Y$ be a smooth map between smooth manifolds that is one-to-one on a compact submanifold Z of X. Suppose that for all $x \in Z$

$$df_x : T_x(X) \to T_{f(x)}(Y)$$

is an isomorphism. Prove that there is a neighborhood U of Z such that

$$f|U : U \to f(U)$$

is a diffeomorphism.

3. Identify the set of all $n \times n$ matrices $M(n)$ with real entries with $\mathbb{R}^{n^2}$ and let

$$SL_n(\mathbb{R}) = \{M \in M(n) | \det(M) = 1\}$$

a) Show that $SL_n(\mathbb{R})$ is a submanifold of $M(n)$.
b) Compute the tangent space $T_I(SL_n(\mathbb{R}))$ at the identity matrix.

4. Let X be any subset of $\mathbb{R}^n$ and $f : X \to \mathbb{R}$ a function with the following property. For every $x \in X$ there is a neighborhood U_x of x in $\mathbb{R}^n$ and a smooth function $f_x : U_x \to \mathbb{R}$ such that $f = f_x$ on $X \cap U_x$. Show that there is an open set U in $\mathbb{R}^n$ containing X and a smooth function $g : U \to \mathbb{R}$ such that $g|X = f$.

5. Consider the 1-form on $\mathbb{R}^2 \setminus \{0\}$

$$\omega = \frac{xdy - ydx}{x^2 + y^2}$$

a) Prove that ω is closed, i.e. $d\omega = 0$.

b) Prove that ω is not exact, i.e. there is no smooth function $f : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ such that $\omega = df$.

6. Give a proof of the Boundary Theorem: If a smooth map $f : X \to Y$ between two closed oriented n-manifolds extends to a smooth map $F : W \to Y$ of a compact oriented $(n+1)$-manifold W with $\partial W = X$ then $\deg f = 0$.

7. Regard the real projective plane $\mathbb{R}P^2$ as the space of triples $(x, y, z) \in \mathbb{R}^3 \setminus \{0\}$ modulo the relation $(x, y, z) \sim (tx, ty, tz)$ for $t \in \mathbb{R} \setminus \{0\}$. Define

$$f : \mathbb{R}P^2 \to \mathbb{R}$$

by

$$f(x, y, z) = \frac{x^2 + 2y^2}{x^2 + y^2 + z^2}$$

Compute all critical points of f. Show that f is a Morse function and compute the Morse index of each critical point.

8. Consider the following vector fields in $\mathbb{R}^3$:

$$X = \frac{\partial}{\partial x}, \quad Y = x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

Here x, y, z are the coordinates on $\mathbb{R}^3$ and $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ are the three standard coordinate vector fields. Show that there is no (nonempty) surface $M \subset \mathbb{R}^3$ which is tangent to both vector fields at each of its points.

9. Show that every map $f : S^2 \to S^2$ from the 2-sphere to itself admits a point $x \in S^2$ such that $f(x) = \pm x$.

10. Let Σ be an oriented genus 2 surface smoothly embedded in $\mathbb{R}^3$.

a) Describe the Gauss map $g : \Sigma \to S^2$ and define the Gaussian curvature at a point of Σ in terms of g.

b) State the Gauss-Bonet theorem for this situation.

c) Prove that Σ has (many) saddle points, i.e. points of negative curvature.

B. Answer five of the following questions. Each question is worth ten points.

11. Compute the fundamental group of the space obtained from two tori $S^1 \times S^1$ by identifying a circle $S^1 \times \{x_0\}$ in one torus with the corresponding circle $S^1 \times \{x_0\}$ in the other torus.

12. Give an explicit example of a covering space of $X = S^1 \vee S^1$ which is not normal.

13. Compute the homology groups of the space obtained from a torus $S^1 \times S^1$ by attaching a Möbius band via a homeomorphism from the boundary circle of the Möbius band to the circle $S^1 \times \{x_0\}$ in the torus.

14. Suppose M is a closed connected 5-manifold such that $\pi_1(M) \cong \mathbb{Z}/3\mathbb{Z}$. If $H_2(M;\mathbb{Z}) \cong \mathbb{Z}$, compute all other homology and cohomology groups of M with integral coefficients.

15. a) Describe the (simplest) cell structure of $\mathbb{R}P^n$ (give the attaching maps of all cells – you don't have to give a proof here).

b) Write down the cellular chain complex associated with this cell decomposition.

c) Use this chain complex to compute homology and cohomology groups of $\mathbb{R}P^n$ with coefficients in $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$.

16. Let $p : \tilde{X} \to X$ be a 2-sheeted covering map between cell complexes such that the cell structure on $\tilde{X}$ is obtained by lifting the cell structure on X.

a) Construct a short exact sequence of cellular chain complexes with coefficients in $\mathbb{Z}/2\mathbb{Z}$

$$0 \to C_n(X;\mathbb{Z}/2\mathbb{Z}) \to C_n(\tilde{X},\mathbb{Z}/2\mathbb{Z}) \to C_n(X,\mathbb{Z}/2\mathbb{Z}) \to 0$$

(i.e. give the maps explicitly and prove that the sequence is exact).

b) Write down the induced long exact sequence of homology groups (the transfer sequence).

c) Prove that if $\tilde{X}$ is acyclic over $\mathbb{Z}/2\mathbb{Z}$ (i.e. $H_i(\tilde{X};\mathbb{Z}/2\mathbb{Z}) = 0$ for $i > 0$ and $H_0(\tilde{X};\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$) then $H_i(X;\mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$ for all $i \geq 0$.

17. Compute $\pi_2(S^1 \vee S^2)$ using the Hurewicz theorem.

18. Let $p : S^3 \to \mathbb{C}P^1 = S^2$ be the map that sends $(z, w) \in S^3 \subset \mathbb{C}^2$ to its equivalence class $[z : w] \in \mathbb{C}P^1$.

a) Prove that p is a fiber bundle by explicitly writing down local trivializations.

b) Write down the associated long exact sequence of homotopy groups.

c) Compute $\pi_3(S^2)$. Justify all auxiliary results you are using.

19. Let F_n be a free group of rank n and let $G \subset F_n$ be a subgroup of index m. Prove that G is a free group and compute its rank.

20. Let Σ be a closed oriented surface of genus g and suppose that $X \subset \Sigma$ is a graph which is a retract of Σ. Prove that

$$rank H_1(X) \leq g$$

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF UTAH
Ph.D. PRELIMINARY EXAMINATION IN GEOMETRY/TOPOLOGY
August 12, 2002, 9:00-12:00

Instructions: Do five (5) problems from section A and five (5) problems from section B. Be sure to provide all relevant definitions and statements of theorems cited. Make sure you indicate which solutions are to be graded, otherwise the first five answered will be scored.

A. Answer five of the following questions. Each question is worth ten points.

1. Prove that no compact differentiable manifold (without boundary), except for the point, is contractible.

2. Consider a compact differentiable m-manifold M without boundary and let $N = M \times M$. Let $\Delta \subset N$ be the diagonal. Prove that there is no compact differentiable $m+1$-dimensional submanifold with boundary $B \subset N$, so that the boundary of B is Δ.

3. State and prove Brouwer's fixed-point theorem.

4. Prove that the Moebius band is not contractible.

5. Using methods of differential topology prove that for each compact orientable differentiable manifold (without boundary) M of dimension m, $H^m(M, \mathbb{R}) \neq 0$, where $H^m(M, \mathbb{R})$ is the de Rham cohomology group.

6. True or false (provide justification): Let $M \subset \mathbb{R}^n$ be a smooth submanifold of dimension m. Then there are $n-m$ smooth functions $f_1, ..., f_m : \mathbb{R}^n \to \mathbb{R}$ such that:

a. $M = \{x \in \mathbb{R}^n : f_1(x) = ... = f_{n-m}(x) = 0\}$.

b. For each $x \in M$ the gradients $\nabla(f_1)(x), ..., \nabla(f_{n-m})(x)$ are linearly independent.

7. Give definition of a (differentiable) orientable manifold. Prove that product of two orientable manifolds is again orientable.

8. Give definition of the tangent bundle of a smooth n-dimensional manifold. Prove that the tangent bundle is also a smooth manifold.

9. Compute the pull-back of the differential form $\omega = xdx \wedge dy +$

$zdy \wedge dz$ in $\mathbb{R}^3$ under the map $f : \mathbb{R}^2 \to \mathbb{R}^3$ given by

$$f(u,v) = (u^2, u+v, \sin(u+v)).$$

10. Define the index of a vector field on a smooth manifold. Construct an explicit vector field on the 2-sphere S^2 and compute the Euler characteristic of S^2 by computing the index of this vector field.

B. Answer five of the following questions. Each question is worth ten points.

Let M and N be two closed (that is compact without boundary) n-dimensional manifolds. Denote by M_0 (resp. N_0) the complement of an embedded open n-disk in M (resp. N). Note that $\partial M_0 \approx \partial N_0 \approx S^{n-1}$. Recall that *the connected sum* $M\#N$ is the result of gluing of M_0 and N_0 along their boundary.

11. Express the Euler characteristic $\chi(M\#N)$ in terms of $\chi(M)$ and $\chi(N)$.

12. Express the fundamental group $\pi_1(M\#N)$ in terms of $\pi_1(M)$ and $\pi_1(N)$ if $n > 1$.

13. Prove that $M\#N$ is orientable if and only if both M and N are orientable.

14. Prove that the Euler characteristic of any closed odd-dimensional manifold is zero.

15. The 4-dimensional cube $[0,1] \times [0,1] \times [0,1] \times [0,1]$ has 16 vertices, 32 edges and 24 2-dimensional faces. How many 3-dimensional faces does it have?

16. Find a degree 2 map from $S^1 \times S^1$ to S^2.

17. Let Σ be a simply-connected closed 3-manifold. Prove that $H_k(\Sigma) = H_k(S^3)$ for any integer k.

18. Describe all possible ways in which the homology of a space can change after attaching an n-cell.

19. Let X be a simply-connected closed 4-manifold. Prove that $H_k(X)$ is torsion-free for any integer k.

20. Let X be a topological space such that $H_0(X) = \mathbb{Z}$, $H_1(X) = 0$, $H_2(X) = \mathbb{Z}_2$, $H_3(X) = \mathbb{Z}_3$, $H_4(X) = \mathbb{Z}$, $H_k(X) = 0$, $k > 4$. Compute $H_*(X;\mathbb{Z}_2)$ and $H^*(X;\mathbb{Z})$.

DEPARTMENT OF MATHEMATICS
University of Utah
Ph.D. PRELIMINARY EXAMINATION IN GEOMETRY/TOPOLOGY
August 14, 2001, 9:00-12:00, JWB 333

Instructions: Do five (5) problems from section A and five (5) problems from section B. Be sure to provide all relevant definitions and statements of theorems cited. Make sure you indicate which solutions are to be graded, otherwise the first five answered will be scored.

A. Answer five of the following questions. Each question is worth ten points. All manifolds are differentiable.

1. Let X and Y be two submanifolds of $\mathbb{R}^n$. Show that there is $a \in \mathbb{R}^n$ such that $X + a = \{x + a | x \in X\}$ and Y are transverse.

2. Recall that an $n \times n$ real matrix M is *orthogonal* if

$$< Mv, Mw >=< v, w >$$

for all $v, w \in \mathbb{R}^n$ (and $< \cdot, \cdot >$ is the inner product). Let $O(n)$ denote the set of all orthogonal $n \times n$ matrices, viewed as a subset of $\mathbb{R}^{n^2}$. Show that $O(n)$ is a manifold, and compute its dimension.

3. By $a : S^n \to S^n$ denote the antipodal map. Then the space $A^p(S^n)$ of smooth p-forms on S^n breaks up into the $+1$ and the -1 eigenspace of a^*, i.e.

$$A^p(S^n) = A^p_+(S^n) \oplus A^p_-(S^n)$$

where

$$A^p_\epsilon(S^n) = \{\omega \in A^p(S^n) | a^*(\omega) = \epsilon\omega\}.$$

Show that if ω is an exact p-form and belongs to the $+1$-eigenspace, then $\omega = d\eta$ for some $(p-1)$-form η in the $+1$-eigenspace. What can you conclude about DeRham cohomology $H^i(\mathbb{R}P^n)$ from this?

4. a) Give a definition of the tangent bundle TM of a manifold M.

b) Give a definition of what it means for a manifold to be orientable.

c) Show that TM, viewed as a differentiable manifold, is always orientable.

5. Recall that a map $f : X \to \mathbb{R}$ defined on a subset $X \subset \mathbb{R}^n$ is said to be *smooth* if for every $x \in X$ there is an open set $U_x \ni x$ in $\mathbb{R}^n$ and a smooth function $f_x : U_x \to \mathbb{R}$ such that $f = f_x$ on $X \cap U_x$. Prove that if $f : X \to \mathbb{R}$ is smooth then there is a neighborhood U of X and a smooth extension $\tilde{f} : U \to \mathbb{R}$.

6. Define the index of a singularity of a vector field and state the Poincaré-Hopf index theorem.

7. Consider the vector fields

$$X = \frac{\partial}{\partial x} \text{ and } Y = x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}$$

on $\mathbb{R}^3$. Prove or disprove: There is a nonempty surface $M^2 \subset \mathbb{R}^3$ such that for every $p \in M$ the tangent space T_pM is spanned by X_p and Y_p.

8. Let X be a closed manifold, $\Delta = \{(x, x) \in X \times X\}$ the diagonal, and $N(\Delta, X \times X) = \{((x, x), (v, -v)) | x \in X, v \in T_x(X)\}$ the normal bundle of Δ. Show that there is a diffeomorphism of a neighborhood of X (viewed as the zero section) in the tangent bundle $T(X)$ with a neighborhood of Δ in $X \times X$, extending the usual diffeomorphism $X \to \Delta$, $x \mapsto (x, x)$.

9. Give a definition of (Gaussian) curvature for surfaces in $\mathbb{R}^3$ and state the Gauss-Bonnet theorem.

10. Compute the pullback $f^*\omega$ of the differential form

$$\omega = dx \wedge dy$$

by the map $f : \mathbb{R}^3 \to \mathbb{R}^3$, $f(x, y, z) = (xyz, y + x, z)$.

B. Answer five of the following questions. Each question is worth ten points.

11. Let M be a nonempty compact manifold, smooth or triangulated (you can make a choice). Show that M does not admit a retraction to its boundary.

12. Let M be a closed connected 7-manifold such that: $H_1(M; \mathbb{Z}) = 0$, $H_2(M; \mathbb{Z}) = \mathbb{Z}$, $H_3(M; \mathbb{Z}) = \mathbb{Z}/2$. Compute $H_i(M; \mathbb{Z})$ and $H^i(M; \mathbb{Z})$ for all i.

13. Prove that $\mathbb{R}P^3$ and $\mathbb{R}P^2 \vee S^3$ are not homotopy equivalent. (Hint: Cup products.)

14. Let $X \subset \mathbb{R}^3$ be the union of n lines through the origin. Compute $\pi_1(\mathbb{R}^3 \setminus X)$.

15. Let $p : M \to N$ be a covering map between two closed connected manifolds of positive dimension. Show that p is not null-homotopic.

16. a) Define carefully the Hopf map $h : S^3 \to S^2$.

b) Define what it means for a map to be a fiber bundle and sketch a proof that h is a fiber bundle.

c) State the long exact sequence in homotopy groups associated to a fiber bundle and compute $\pi_3(S^2)$.

17. Let X be the space obtained from the torus $S^1 \times S^1$ by attaching a Möbius band via a homeomorphism from the boundary circle in the Möbius band to the circle $S^1 \times \{x_0\}$ in the torus. Compute $\pi_1(X)$ and describe the universal cover of X.

18. Construct carefully a Δ-complex structure on $\mathbb{R}P^2$ and use it to compute homology groups of $\mathbb{R}P^2$ with $\mathbb{Z}$ and with $\mathbb{Z}_2$ coefficients.

19. The surface M_g of genus g, embedded in $\mathbb{R}^3$ in the standard way, bounds a compact region R. Two copies of R, glued together by the identity map between their boundary surfaces M_g, form a closed 3-manifold X. Compute the homology groups of X via the Mayer-Vietoris sequence for this decomposition of X into two copies of R.

20. a) Show that the Klein bottle cannot be embedded (as a

smooth submanifold) into $\mathbb{R}^3$.

b) It is a fact that the Klein bottle can be embedded into $\mathbb{R}^4$. Show that no such embedding $K \subset \mathbb{R}^4$ can be given by a system of equations

$$f_1 = 0, f_2 = 0$$

with independent smooth functions f_1, f_2 (i.e. where df_1, df_2 are linearly independent at each point $x \in K$).

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF UTAH

PH.D. PRELIMINARY EXAM IN GEOMETRY-TOPOLOGY

Fall 2000

Instructions. Do all nine problems. Do all of problems 1,2 and 3. For each of problems 4 through 9, do either part A or part B; make sure to indicate which part is to be graded. All problems are worth 10 points each.

Problem 1. Let Σ be a smooth, compact, orientable surface (2 dimensional manifold) of genus g. Prove that Σ admits a smooth non-vanishing vectorfield if and only if $g = 1$.

Problem 2. Let M be a smooth, orientable n-dimensional manifold. Prove that the n dimensional DeRham cohomology with compact supports, $H^n_c(M)$ is nonzero.

Problem 3. Without resorting to the DeRham Theorem, prove that $H^2(\mathbf{R}^2) = 0$ (where $H^2(\mathbf{R}^2)$ denotes the 2 dimesional DeRham cohomology).

Reminder!. In each of the folowing problems, do either part A, or part B.

Problem 4.

A. Let $S = \{(z_1, \dots, z_n) \in \mathbf{C}^n : z_1^2 + \cdots + z_n^2 = 1\}$. Prove that S is a smooth (real) submanifold of $\mathbf{C}^n = \mathbf{R}^{2n}$, and find it's dimension.

B. Prove that the orthogonal group $O(n)$, is a smooth submanifold of the space of $n \times n$ matrices $M(n) = \mathbf{R}^{n^2}$, and find it's dimension.

Problem 5.

A. Let $f : S^n \to T^n$ be continuous, $n > 1$, where S^n and T^n denote the n-sphere, and the n dimensional torus respectively. Prove that the degree of f is zero. (If you prefer, you may assume that f is smooth).

B. Let $f : X \to Y$ be a covering map between two connected manifolds. Prove that if f is homotopic to a constant map then X is contractible.

Problem 6.

A. Find an explicit CW complex structure for the complex projective space $\mathbf{C}P^3$, and compute the singular homology groups (over $\mathbf{Z}$) using cellular homology.

B. Find an explicit CW complex strucure for $S^2 \times S^4$, and compute the singular homology groups using cellular homology.

Problem 7.

A. Prove the Brouwer Fixed Point Theorem using homology theory. That is, if $f : B^n \to B^n$ is continuous, where B^n denotes the closed unit ball in $\mathbf{R}^n$, then f has a fixed point.

B. Prove the Fundamental Theorem of Algebra using degree theory.

Problem 8.

A. Compute the DeRham cohomology (or if you prefer, the singular homology) of $(S^2 \times S^2)\#\mathbf{C}P^2$, where, $M\#N$ denotes the connected sum of the manifolds M and N.

B. Let p be a positive integer. Compute the singular homology with coefficients in $\mathbf{Z}_p$ of real projective 3-space ($\mathbf{Z}_p$ being the cyclic group of order p). That is, find $H_k(\mathbf{R}P^3, \mathbf{Z}_p)$ for $k \geq 0$.

Problem 9.

A. Let F_g be a closed orientable surface of genus g. Prove that $\pi_1(F_g)$ is infinite, and non-abelian, provided $g > 1$.

B. Let G be the one point union of two circles (a figure eight). Draw a picture of an explicit irregular covering map $\tilde{G} \to G$.

DEPARTMENT OF MATHEMATICS
University of Utah
Ph.D. PRELIMINARY EXAMINATION IN GEOMETRY/TOPOLOGY
August 17, 1999, 9:00-12:00, JWB 333

Instructions: Do five (5) problems from section A and five (5) problems from section B. Be sure to provide all relevant definitions and statements of theorems cited. Make sure you indicate which solutions are to be graded, otherwise the first five answered will be scored.

A. Answer five of the following questions. Each question is worth ten points.

1. Let $f : M \to N$ be a submersion between two smooth manifolds (without boundary).

a) Show that f is an open map, i.e., if $U \subseteq M$ is open, then $f(U) \subseteq N$ is open.

b) Show that if M is closed (i.e. compact without boundary) and N is connected, then f is onto.

c) Show that there is no submersion of a closed manifold to $\mathbb{R}^n$.

2. Let M be a smooth manifold and $p : E \to M$ a smooth vector bundle on M (with fibers finite-dimensional $\mathbb{R}$-vector spaces).

a) Define the notion of a smooth (Euclidean, also called Riemannian) metric on E.

b) Using partitions of unity, show that E admits a smooth metric.

3. Prove that the n-sphere admits a nowhere vanishing vector field if and only if n is odd.

4. Show that $SL_2(\mathbb{R})$, viewed as the subset $\{(a, b, c, d) | ad - bc = 1\}$ of $\mathbb{R}^4$, is a manifold.

5. Consider the two vector fields in $\mathbb{R}^3$ given by $V(x, y, z) = (x^2 + y^2, 0, -y)$ and $W(x, y, z) = (0, x^2 + y^2, x)$. Find a two dimensional surface $M^2 \subset R^3$ containing the point $p = (1, 0, 0)$ which is tangent to the vector fields V and W in a neighborhood of p, or show that such a surface M doesn't exist.

6. A symmetric quadratic form on $\mathbb{R}^2$ is given by $ds^2 = \frac{dx^2+dy^2}{(1+x^2+y^2)^2}$

a) Show that ds^2 is a Riemannian metric for $\mathbb{R}^2$.

b) Compute the length of the curve $t \mapsto (0, t)$ for $a \leq t \leq b$ in this metric.

c) Compute the area of $\mathbb{R}^2$ in this metric.

d) Show that $(x, y) \mapsto \frac{(x,-y)}{x^2+y^2}$ is an isometry on $(\mathbb{R}^2 - \{(0,0)\}, ds^2)$.

7. Let Σ^2 be a (nonempty) smooth closed (compact without boundary) surface smoothly embedded in $\mathbb{R}^3$. Show that there is a two-plane P in $\mathbb{R}^3$ which intersects Σ such that $P \cap \Sigma$ consists of finitely many disjoint smooth closed loops.

8. True or false (give a short explanation or a counterexample): Let Σ be a smooth, closed, oriented surface. Assume that Σ has a Riemannian metric of constant curvature K. Suppose that $\gamma \subset \Sigma$ is a closed embedded geodesic loop which separates Σ. (That is, $\Sigma - \gamma$ has two components.)

a) Then $K \leq 0$.

b) Then $K \neq 0$.

9. Let M^n be a smooth oriented manifold. Suppose $f : M \to \mathbb{R}$ is a smooth function such that $df \neq 0$ at each point of M. Show that there is a smooth vector field V on M so that $df(V) \leq -1$. Show also, that if ϕ_t is the flow generated by V, then $f(\phi_1(x)) \leq f(x) - 1$ for all $x \in M$ such that $\phi_1(x)$ exists.

10. Let ω be a smooth closed p-form on S^n with $p > 0$. Show that there is a smooth $(p-1)$-form α so that $d\alpha = \omega$ if $p < n$ but that this need not be the case when $p = n$. You are allowed to quote standard calculations in deRham cohomology.

B. Answer five of the following questions. Each question is worth ten points.

11. Let $T^2 = S^1 \times S^1$ be the standard 2-torus and $p, q \in T^2$ be two distinct points. Let $X = T^2/\sim$ be the identification space where the two points are identified $p \sim q$. Find $\pi_1(X)$.

12. Let $U(n)$ denote the set of unitary matrices (the complex $n \times n$ matrices A which satisfy $A\bar{A}^T = I$). Show that $U(n-1) \to U(n) \to S^{2n-1}$ is a fiber bundle. (Hint 1: Find explicit local trivializations. Hint 2: State a general theorem whose conclusion is that a smooth map between closed manifolds is a submersion and verify the hypotheses.)

Deduce that $\pi_r(U(n)) \simeq \pi_r(U(n+1))$ for $r < 2n$. What are $\pi_1(U(3))$, $\pi_2(U(3))$, $\pi_3(U(3))$?

13. Find the Euler characteristic of the connected sum $\chi(RP^3\sharp(S^1\times S^2))$.

In #14 and #15 by Σ_g we denote a closed oriented surface of genus g.

14. Show that every map $f : \Sigma_2 \to \Sigma_3$ has degree 0.

15. Prove that $\pi_2(\Sigma_2) = 0$.

16. Prove that $\mathbb{C}P^2$ and the wedge sum (one point union) $S^2 \vee S^4$ are not homotopy equivalent.

17. True or false (give a short explanation or a counterexample): The Klein bottle cannot be embedded in $\mathbb{R}^3$ as a smooth submanifold.

18. Classify all connected covering spaces of $\mathbb{R}P^2 \vee \mathbb{R}P^2$ up to equivalence.

19. Describe all possible ways in which the homology of a space can change after attaching an n-cell.

20. Let M and N be two connected smooth manifolds. Show that $M \times N$ is orientable if and only if both M and N are orientable.

DEPARTMENT OF MATHEMATICS
University of Utah
Ph.D. PRELIMINARY EXAMINATION IN GEOMETRY/TOPOLOGY
Autumn 1998

Instructions: **Do five (5) problems from section A and five (5) problems from section B. Be sure to provide all relevant definitions and statements of theorems cited. Make sure you indicate which solutions are to be graded, otherwise the first five answered will be scored.**

A. Answer five of the following questions. Each question in this section is worth ten points.

(1) Let X be the set of points $(x, y, z) \in \mathbf{R}^3$ satisfying the equation

$$(1+z)x^2 - (1-z)y^2 = 2z(1-z^2).$$

Show that X is a smooth oriented surface embedded in $\mathbf{R}^3$.

(2) For $(x, y, z) \in \mathbf{R}^3$ let $\eta = dx \otimes dx + dy \otimes dy - dz \otimes dz$ denote a symmetric 2-form on $\mathbf{R}^3$. Let $M = \{(x, y, z) \in \mathbf{R}^3 : 1 + x^2 + y^2 = z^2, z > 0\}$ be one nappe of the hyperboloid.

a. Compute $\int_\Omega dx \wedge dy + (4 - 2z)dx \wedge dz + (z - 2)dy \wedge dz$ where $\Omega \subset M$ is the region inside $x^2 + y^2 \leq 3$.

b. Show that the restriction $\eta|_M$ is a Riemannian metric for M.

c. What is the length of the curve $t \mapsto (\sinh t, 0, \cosh t)$ for $a \leq t \leq b$ in this metric?

(3) Suppose the three sphere $\mathbf{S}^3 = \{(x, y, z, w) \in \mathbf{R}^4 : x^2 + y^2 + z^2 + w^2 = 1\}$ is embedded into four space the standard way. Two independent vector fields on $\mathbf{S}^3$ are given by $V = (y, -x, w, -z)$, $W = (w, z, -y, -x)$. Determine whether it is possible to embed a closed smooth two dimensional surface $N \subset \mathbf{S}^3$ so that the vector fields are tangent to N at every point of N, *i.e.* $V(p), W(p) \in T_pN$ for every $P \in N$.

(4) a. Show that $\Phi : \mathbf{R}^2 \to \mathbf{R}^2$ given by $(p, q) = \Phi(x, y) = (x + y^2 + 2y, x + y^2 - 2y)$ is a diffeomorphism. Compute $d\Phi(\partial/\partial y)$ and the pullback $\Phi^*(p^2\, dp \wedge dq)$.

b. Let X be a smooth vector field and ω a smooth n-form on the smooth compact boundaryless manifold M^n. Let $M \times \mathbf{R} \ni (x, t) \mapsto \phi_t(x) \in M$ be the flow generated by X. Let $A(t) = \int_M {\phi_t}^*\omega$ be the integral of the pulled back n-form. Find dA/dt.

(5) Let $M^m, N^n \subset P^p$ be smooth embedded submanifolds of a smooth manifold. M^m and N^n are said to *intersect transversally* if $T_xM + T_xN = T_xP$ for every $x \in M \cap N$. Prove that if M^m and N^n intersect transversally then $M^m \cap N^n$ is a smooth submanifold of P^p. What is its dimension?

(6) Let $X = \mathbf{S}^1 \times \mathbf{R}$. Find the de Rham cohomology of X in two ways.

a. Using de Rham's Theorem.

b. Directly from the definition.

(7) Let $f : \mathbf{S}^n \to \mathbf{S}^n$ be a continuous map of spheres whose degree $D(f) \neq 1$. Show that there is a point $x \in \mathbf{S}^n$ such that $f(x) = a(x)$ where a is the antipodal map.

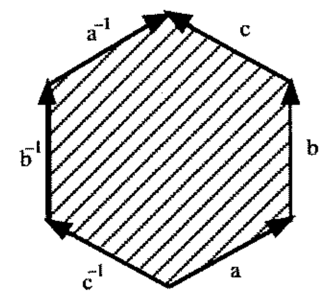
(8) Suppose that $K_1 \supseteq K_2 \supseteq K_3 \supseteq \cdots$ is a nested sequence of compact connected subsets of $\mathbf{S}^2$ such that $\pi_1(K_j) = \mathbf{Z}$ for all j. Prove or provide a counterexample for each of the following statements:

a. $\bigcap_{j=1}^{\infty} K_j$ is connected.

b. $\pi_1(\bigcap_{j=1}^{\infty} K_j) \cong \mathbf{Z}$.

B. Answer five of the following questions. Each question in this section is worth ten points.

(9) Let X be the space obtained from a disk by identifying the boundary according to the following diagram. Find the fundamental group $\pi_1(X)$. Is X homeomorphic to a closed surface? Explain.



(10) Determine all covering spaces of the two-torus $\mathbf{T}^2$ up to equivalence. Explain.

(11) Let $\mathbf{t}^2 \subset \mathbf{S}^5$ be a homeomorph of the two torus $\mathbf{T}^2$. Find the singular homology $H_*(\mathbf{S}^5 - \mathbf{t}^2; \mathbf{Z})$.

(12) Let $X_p = \mathbf{D}^4 \cup_f \mathbf{S}^3$ be the space obtained by attaching a 4-cell to the 3-sphere by an attaching map $f : \partial\mathbf{D}^4 \to \mathbf{S}^3$ of degree p.

a. Compute the singular homology of $H_*(X_p; \mathbf{Z})$ and $H_*(X_p \times X_q; \mathbf{Z})$ for any (p, q) using the CW-complex cell structure.

b. Find $H_*(X_p \times X_q; \mathbf{Z})$ using the Künneth Theorem.

(13) Find the singular homology groups $H_*(\mathbf{RP}^5; \mathbf{Z})$.

(14) Explicitly describe a nontrivial homotopy class in $\pi_3(\mathbf{S}^2)$. Show that it is nonzero.

(15) Suppose that (C_k, ∂_k) is a free chain complex. Suppose that it is *acyclic*, *i.e.* the homology groups $H_k(C) = 0$ for all k. Show that C is *contractible*, that is, the identity map and the zero map $\iota_k, \zeta_k : C_k \to C_k$ are chain homotopic.

(16) Prove that $\mathbf{CP}^2$ cannot be decomposed as a union $M_1 \cup M_2$ where M_1 and M_2 are homeomorphic compact submanifolds with boundary and $M_1 \cap M_2$ is the common boundary of M_1 and M_2.

(17) Let $X = \mathbf{S}^1 \vee \mathbf{S}^2 \vee \mathbf{T}^2$ be the one point union. Compute $\pi_1(X)$ and show that $\pi_2(X)$ is infinitely generated.

(18) Let $X = \Delta - \cup_{j=1}^{\infty} D_j^\circ$ where Δ is the closed unit disk and $D_j^\circ \subset \Delta$ are open round disks with pairwise disjoint closures. Prove that the homology of X is infinitely generated.

Ph. D. preliminary examination in Geometry/Topology

September 22, 1997

Instructions: Do the indicated number of problems from each of the two sections below. The problems in Section I are worth 10 points each. The problems in Section II are worth 15 points each, be sure to provide all relevant definitions and statements of theorems cited. Make sure you indicate which solutions are to be graded, otherwise the first problems you have answered will be scored.

Section I. Pick six of the following ten questions to answer:

Problem 1. Show that $\pi_1(\mathbb{RP}^2\#\mathbb{RP}^2\#\mathbb{RP}^2)$ is infinite.

Problem 2. Using homology prove Brouwer's fixed-point theorem:
If $f : I^n \to I^n$ is a continuous map then there exists $x \in I^n$ such that $f(x) = x$. (Here and below I denotes the closed unit interval.)

Problem 3. Prove the fixed-point theorem for contracting maps:
Let X be a complete metric space and $f : X \to X$ be a map such that there exists $\lambda < 1$ so that

$$d(f(x), f(y)) \leq \lambda d(x, y)$$

for all $x, y \in X$. Then f has a fixed point.

Problem 4. Prove that $\mathbb{R}^n$ is not homeomorphic to $\mathbb{R}^k$ for $k \neq n$. (Do not appeal to Brouwer's invariance of domain theorem.)

Problem 5. True of false:
There is no orientable smooth 3-dimensional manifold M and a smooth embedding $\mathbb{RP}^2 \to M$. Justify your answer.

Problem 6. Let $p(z) = a_n z^n + ... + a_0$ be a non-constant complex polynomial. Prove $p : \mathbb{C} \to \mathbb{C}$ is proper, has only a finite number of singular values, that the inverse images of regular values have constant cardinality. Deduce the Fundamental Theorem of Algebra: p has a root.

Problem 7. Give a proof or a counterexample to the following assertion:
If $H_n(X; \mathbb{Z}) = 0$ then $H_n(X; G) = 0$ for any abelian group G.

Problem 8. Let $M^n \subset \mathbb{R}^{n+1}$ be an embedded differentiable submanifold. Prove that for each $P \in M$ there is a neighborhood $U \subset \mathbb{R}^{n+1}$ of P so that $U \cap M$ is expressible as the graph "above" at least one of the $n + 1$ coordinate hyperplanes of $\mathbb{R}^{n+1}$.

Problem 9. Let $f, g : K \to L$ be two chain maps ($f_n, g_n : K_n \to L_n$). Define what it means for f and g to be chain homotopic. Prove that if f and g are chain homotopic, then they induce the same maps in homology, $f_* = g_* : H_n(K) \to H_n(L)$.

Problem 10. Identify $\mathbb{R}^4$ with $\mathbb{C}^2$ so that $z = x_1 + iy_1, w = x_2 + iy_2$. Let $p = (1, 0) \in \mathbb{C}^2$. Consider the following smooth submanifolds of $\mathbb{C}^2 - \{(0,0), (0,0.5)\}$:

$$M_1 := \{(z,w) | z^2 + w^2 = 1\}$$

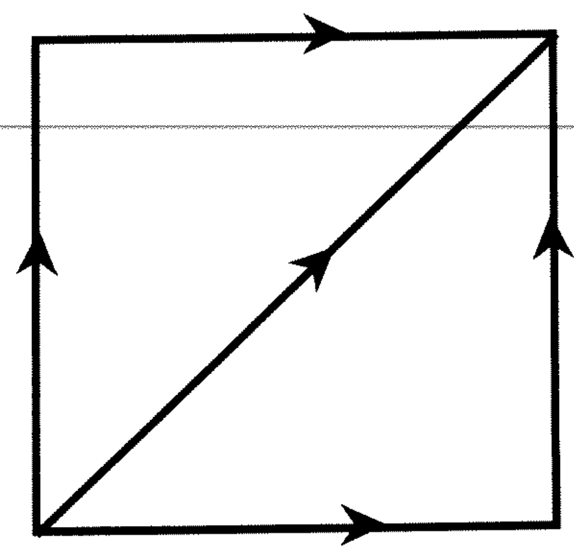
$$M_2 := \{(z,w) | z^2 - w^2 + w = 1\}$$

$$M_3 := \{(x_1, y_1, x_2, y_2) | 1 - x_1 + x_2^2 + y_1^2 + y_2^2 = 0\} .$$

Which of the pairs of manifolds (M_1, M_2), (M_2, M_3), (M_1, M_3) are transverse at p?

Section II. Pick four of the following eight questions to answer:

Problem 1. Let X be the topological space described on the Figure below: it is obtained from the 2-dimensional square I^2 by identifying segments with arrows (the identification respects the orientation of edges determined by the arrows). Compute $\pi_1(X)$.



X

Problem 2. True or false:

Any closed differential form of degree ≥ 1 on $\mathbb{RP}^2 \times \mathbb{RP}^2$ is exact. Justify your answer.

Problem 3. Let S^4 be the 4-dimensional sphere and $F \subset S^4$ be a smoothly embedded surface of genus g (this surface is orientable, compact and has no boundary). Compute the homology groups

$$H_k(S^4 - F, \mathbb{R})$$

for $k = 2, 3$. (You may assume that F is contained in an open subset U of $\mathbb{R}^4$, which is homeomorphic to $F \times \mathbb{R}^2$ so that $F \subset U$ corresponds to $F \times \{0\}$.)

Problem 4. Consider the set of matrices

$$O(n) = \{A \in \mathcal{M}_n : AA^T = 1\}$$

where $\mathcal{M}_n \cong \mathbb{R}^{n^2}$ is the collection of all $n \times n$-matrices and A^T means the transposition of A. Show that $O(n)$ is a smooth submanifold of $\mathcal{M}_n$. Describe the set of matrices which form the tangent space to $O(n)$ at the identity matrix.

Problem 5a. Let S^3 be the 3−sphere, and let $T^2 = S^1 \times S^1$ be the 2−torus. Compute the singular homology of $S^3 \times T^2$ using its CW-complex structure.

b. Recompute the singular homology of $S^3 \times T^2$ using the Eilenberg-Zilber and Künneth Theorems.

Problem 6. Define what a Riemannian manifold is. Define the natural distance function which one obtains from the (infinitesimal) Riemannian metric, and prove that this distance function gives a metric space structure to each Riemannian manifold.

Problem 7. Let $X = \{(x, y, z) \in \mathbb{R}^3 | xyz = 0\}$, i.e. X is the union of the three coordinate planes in $\mathbb{R}^3$. Let $0 \in X$ be the origin. Prove that any homeomorphism of X must fix 0.

Problem 8a. Let $P : M \to N$ be a regular covering map between connected differentiable manifolds M, N and let M be orientable. Prove that N is orientable if and only if every deck transformation (covering space automorphism) preserves the orientation of M.

b. Prove that $\mathbb{RP}^n$ is orientable if and only if n is odd.

Ph. D. preliminary examination in Geometry/Topology

September 10, 1996

Instructions: Do the indicated number of problems from each of the two sections below. The problems in Section I are worth 10 points each. The problems in Section I are worth 15 points each, be sure to provide all relevant definitions and statements of theorems cited. Make sure you indicate which solutions are to be graded, otherwise the first problems you have answered will be scored.

Section I. Pick six of the following ten questions to answer:

Problem 1. Let M be the Moebius band which has Riemannian metric with totally-geodesic boundary and curvature $K(p) \leq 0$ in all points $p \in M$. Show that the curvature of this metric is identically zero.

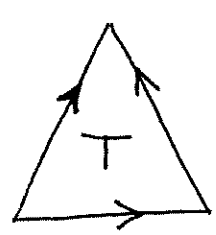
Problem 2. Consider the space $\mathcal{L}$ of all straight lines $ax + by + c = 0$ in the Euclidean plane $\mathbb{R}^2$. Topologize $\mathcal{L}$ so that a sequence of lines L_i is convergent to a line $L = \{ax + by + c = 0\}$ if and only if L_i can be represented by equations $a_i x + b_i y + c_i = 0$ so that $\lim_i a_i = a$, $\lim_i b_i = b$, $\lim_i c_i = c$. Prove that the topological space $\mathcal{L}$ is homeomorphic to the Moebius band without boundary.

Problem 3. Let $F : \mathbb{R}^2 \to \mathbb{R}^3$ be a mapping given by the formula: $F(u, v) = (e^u \cos(v), e^u \sin(v), 0)$. Compute the pull-back $F^*(\omega)$ of the differential form $\omega = xydx \wedge dy + xdy \wedge dz \in \Omega^*(\mathbb{R}^3)$.

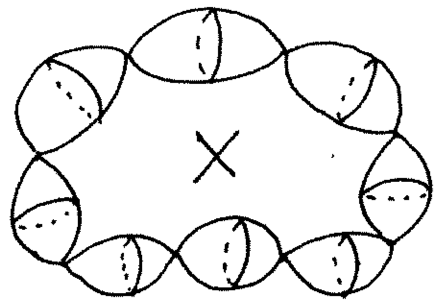
Problem 4. Construct a nontrivial circle bundle over the Klein bottle.

Problem 5. Consider the complex plane $\mathbb{C}$ with the Euclidean metric. Let $f : \mathbb{C} \to \mathbb{C}$ be a complex-analytic mapping and $z \in \mathbb{C}$ be a point such that $f'(z) \neq 0$. Show that the derivative $Df_z : T_z\mathbb{C} \to T_{f(z)}\mathbb{C}$ preserves angles.

Problem 6. Take the 2-dimensional triangle T and identify all its sides as indicated below. What is the universal covering of the resulting topological space X?



Problem 7. Let $n > 2$. By $\mathbb{Z}/n$ denote the cyclic group of order n with discrete topology and by $\mathbb{S}^2$ the 2-sphere with standard topology. Let p and q be two distinct points of $\mathbb{S}^2$. Compute the homology groups of the *n-link sausage space* X which is the space that can be represented as the identification space $(\mathbb{S}^2 \times \mathbb{Z}/n)/\sim$ with the equivalence relation induced by $(p, i) \sim (q, i+1)$.



Problem 8. Denote by $[X, Y]$ the set of free homotopy classes of maps $X \to Y$. Show that $[\mathbb{S}^2, \mathbb{RP}^2]$ is infinite. In other words, construct a sequence of maps $f_i : \mathbb{S}^2 \to \mathbb{RP}^2$, $i = 1, 2, 3, \cdots$ such that no two are homotopic to each other.

Problem 9. Let S be a surface with Riemannian metric whose curvature $K(p)$ is nonpositive at all points p of S. Let $\gamma : \mathbb{R} \to S$ be a geodesic such that $\gamma(t) = \gamma(t+1)$ for all t and γ represents the trivial element of the fundamental group $\pi_1(S, \gamma(0))$. Prove[1] that the mapping γ is constant.

Problem 10. Give an explicit construction of the Hopf fibration $h : \mathbb{S}^3 \to \mathbb{S}^2$ and argue that h is not null-homotopic.

Section II. Pick four of the following eight questions to answer:

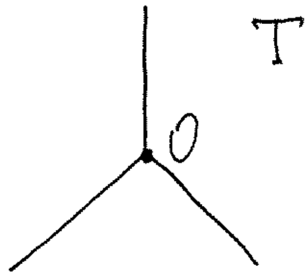
Problem 1. Let $f : \mathbb{S}^n \to \mathbb{S}^n$ be a continuous map from the n-sphere to itself and assume that n is even. Show that there is a point $x \in \mathbb{S}^n$ such that $f(x) \in \{x, -x\}$. Does the same assertion hold if n is odd? If it does, prove it, and if it does not, find a counterexample.

Problem 2. Let S_g denote the closed[2] connected orientable surface of genus g. Suppose that $f : S_m \to S_n$ is a continuous map of degree ± 1.

- Show that $f_{\#} : \pi_1(S_m) \to \pi_1(S_n)$ is onto.
- Conclude that $m \geq n$.

[1] Without referring to Cartan-Hadamard theorem.
[2] I.e. compact and without boundary.

Problem 3. Consider the "tripod" T with the central point O (i.e. the union of three segments in the plane which intersect only at the origin O). Let $P = (O, O)$ be the point on $X = T \times T$ which projects to the point O in the both factors. Compute $H_1(X, X - P; \mathbb{Z})$.



Problem 4. True or false:

(a) Every local homeomorphism between closed connected manifields is a covering onto its image.

(b) Every local homeomorphism between open[3] connected manifields is a covering onto its image.

Justify your answers.

Problem 5. True or false:

Suppose that X is a finite CW-complex and $A \subset X$ is a subcomplex. Then the relative homotopy groups $\pi_j(X, A)$ are isomorphic to the homotopy groups $\pi_j(X/A)$, where X/A is the CW-complex obtained from X by collapsing A to a point.

Justify your answer.

Problem 6. Suppose that the curvature and torsion of a regular curve in $\mathbb{R}^3$ are non-zero constants a, b. Find a parameterization of the curve by arclength.

Problem 7. Let

$$X(s, \theta) = (\varphi(s)\cos(\theta), \varphi(s)\sin(\theta), \gamma(s))$$

be a surface of revolution such that $\varphi'(s)^2 + \gamma'(s)^2 = 1$. Prove that the meridians $\theta = const$ are geodesics. What condition on the lattitude $s = const$ is necessary and sufficient to made it geodesic?

Problem 8. Prove that torus can not be immersed in $\mathbb{R}^3$ in such a way that the Gauss curvature of the induced metric is everywhere non-negative.

[3] I.e. noncompact and without boundary.

DEPARTMENT OF MATHEMATICS
University of Utah
Ph. D. PRELIMINARY EXAMINATION IN TOPOLOGY/GEOMETRY
Autumn 1995

Instructions: Do the indicated number of problems from each of the two parts below (six from Part I, four from Part II). The problems in Part I are worth 10 points each. The problems in part II are worth 15 points each. Be sure to provide all relevant definitions and statements of theorems cited. Make sure you indicate which solutions are to be graded, otherwise the first problems you have answered will be scored.

I. Pick six of the following ten questions to answer

A. Let S_1 and S_2 be two (2-dimensional) surfaces. Derive the relationship between the Euler characteristic of the connected sum $S_1 \sharp S_2$ and the Euler characteristics of S_1 and S_2.

B. Suppose that X is a topological space, $A \subset X$ is a deformation retract which is path connected. Prove that X is path-connected as well.

C. Let G be a subgroup of index 3 in the free group of rank 2. Prove that G is free and find its rank.

D. Suppose M is a connnected manifold. Calculate $H_1(M \times S^1)$ in terms of the homology groups of M.

E. Find a cell-complex structure for the Klein bottle.

F. Suppose that $p : \tilde{X} \to X$ is a covering map between connected manifolds. Prove that if p is null-homotopic, then $\tilde{X}$ is contractible.

G. Show that every continuous map $f : S^n \to S^n$ $(n \geq 1)$ of degree 2 has a fixed point.

H. Let Δ be the 2−dimensional distribution in $\mathbb{R}^3$ defined by $\Delta = \text{span}\{\frac{\partial}{\partial x^1} + x^2\frac{\partial}{\partial x^3}, \frac{\partial}{\partial x^3} + x^3\frac{\partial}{\partial x^2}\}$. Determine whether this distribution is tangent to a family of two-dimensional surfaces.

I. Let (M^n, g) be a Riemannian manifold. Show that the form given locally by $\omega = \sqrt{g}\, dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n$ represents a globally-defined, nowhere vanishing $n-$form. (Here $\sqrt{g}$ is the standard shorthand for the square root of the determinant of the matrix $[g_{ij}] = [< \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} >]$ which gives the metric g in local coordinates.)

J. Calculate $\pi_2(\mathbb{R}P^3)$.

II. Pick four of the following seven questions to answer

1. Let $M_2(\mathbb{R})$ be the set of all 2×2 matrices with real coefficients, and identify it with $\mathbb{R}^4$ in the usual way. Let $SL(2, \mathbb{R})$ be the subset of $M_2(\mathbb{R})$ consisting of those matrices with determinant equal to one.

(a) Show $SL(2, \mathbb{R})$ is a submanifold of $M_2(\mathbb{R})$.

(b) Exhibit the tangent space to $SL(2, \mathbb{R})$ at the identity matrix.

2. Let $f : M \to N$ and $g : N \to M$ be smooth maps, where M and N are compact connected smooth oriented manifolds of the same dimension. If $g \circ f$ is a diffeomorphism, either prove that both f and g are diffeomorphisms, or find a counterexample.

3. Let the curve $(f(s), g(s))$ be parameterized by arclength in $\mathbb{R}^2$ $(f > 0)$. Consider the surface of revolution S in $\mathbb{R}^3$ given parametrically by

$$X(s, \theta) = (f(s)cos\theta, f(s)sin\theta, g(s)).$$

(a) Compute the mean curvature and Gauss curvature for points on S.

(b) Find necessary and sufficient conditions for the "latitudes" obtained when s is constant to be geodesics.

4. Consider the domain D in $\mathbb{R}^3$ which is the complement to the union of two unit balls with centers at the points $(0, 0, 2)$ and $(0, 2, 0)$. Calculate $H_2(D, \mathbb{Z})$.

5. In the product $S^1 \times S^1$ consider the subset

$$A := S^1 \times \{p\} \cup \{p\} \times S^1$$

where p is a point on S^1. Calculate $H_j(A, \mathbb{Z})$, $j = 0, 1, 2$.

6. Prove that the Riemannian connection on a Riemannian manifold (M^n, g) is uniquely determined. (Recall that a Riemannian connection is an affine connection which has zero torsion and is metric compatible.)

7. Let M be a differentiable manifold, ω a closed p-form satisfying

$$\int_C \omega = 0$$

for all smooth p-cycles c. Prove ω is exact.

DEPARTMENT OF MATHEMATICS
University of Utah
Ph.D. PRELIMINARY EXAMINATION IN GEOMETRY/TOPOLOGY
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Instructions: Do eight (8) problems from section I. Do eight (8) problems from section II. Be sure to provide all relevant definitions and statements of theorems cited. Make sure you indicate which solutions are to be graded, otherwise the first eight answered will be scored.

I. Five point questions: Prove if true. Give a counterexample if false.

A. Let A and B be disjoint closed subsets of a topological space X. Then there exists a continuous function $f : X \to [0,1]$ so that $f(A) \equiv 0$ and $f(B) \equiv 1$.

B. Let $K \subset \mathbf{S}^3$ be a knot (a homeomorph of $\mathbf{S}^1$) . Then $\pi_1(\mathbf{S}^3 - K, x_0) = \mathbb{Z}$.

C. $\pi_k\Big(\overbrace{\mathbf{S}^1 \times \cdots \times \mathbf{S}^1}^{n \text{ times}}, x_0\Big) = \begin{cases} \mathbb{Z}^n, & \text{if } k = 1; \\ 0, & \text{if } k > 1. \end{cases}$

D. Let K denote a finite simplicial complex. The Euler characteristic is given by $\chi(K) = \sum_{j \geq 0} (-1)^j \alpha_j$ where α_j is the number of j-simplexes in K. If $\mathbf{Sd}\, K$ denotes barycentric subdivision then $\chi(K) = \chi(\mathbf{Sd}\, K)$.

E. If X and Y are compact connected smooth n-manifolds then for $0 < k < n$, the singular homology satisfies $H_k(X \times Y, \mathbb{Z}) = H_k(X, \mathbb{Z}) \oplus H_k(Y, \mathbb{Z})$.

F. $\mathbf{U}(n)$, the space of complex $n \times n$ unitary matrices, has the structure of a smooth real manifold.

G. Let X be a finite connected graph. Then the fundamental group $\pi_1(X, x_0)$ is a free group with $1 - \chi(X)$ generators. (χ is the Euler characteristic.)

H. Let $\{C_n, d_n\}$ and $\{C'_n, d'_n\}$ be chain complexes of abelian groups and $f_n : C_n \to C'_n$ a chain homotopy equivalence. Then the induced map $f_{n*} : H_n(C) \to H_n(C')$ is isomorphism in homology.

I. Suppose a smooth oriented manifold M^n has a Riemannian metric which is given in any local coordinate system by $ds^2 = \sum_{i,j=1}^n g_{ij}(x)\, dx^i \otimes dx^j$. Then the form given locally in the same coordinate systems by $\omega = \det(g_{ij}(x))\, dx^1 \wedge \cdots \wedge dx^n$ represents a globally defined nowhere vanishing n-form on M.

J. Let $f \in C^\infty(\mathbf{S}^n)$ be a smooth function and let $x_1, x_2 \in \mathbf{S}^n$ be two points such that $f(x_1) < 0 < f(x_2)$. Let ω be a smooth nonvanishing n-form. Then there is a diffeomorphism $\varphi : \mathbf{S}^n \to \mathbf{S}^n$ such that $\int_{\mathbf{S}^n} (\varphi^* f)\, \omega = 0$.

II. Ten point questions.

(1) Let Σ denote the surface obtained from an annulus by identifying antipodal points of the inner circle and by identifying antipodal points of the outer circle. Compute the fundamental group of Σ. Can you identify Σ as a connected sum of tori and projective planes?

(2) Let X be a connected, locally arcwise connected, semilocally simply connected, compact metric space. Show that $\pi_1(X, x_0)$ is finitely generated.

(3) Let $\mathbf{S}^n$ denote the unit sphere of $\mathbb{R}^{n+1}$ and let $a : \mathbf{S}^n \to \mathbf{S}^n$ denote the antipodal map $a(x) = -x$. Show that $\mathbf{S}^n$ admits a nonvanishing tangent vector field if and only if a is homotopic to the identity. For which values of n does this happen?

(4) Find the singular homology $H_*(\mathbf{S}^5 - (A \cup B), \mathbb{Z})$ where A is a homeomorph of $\mathbf{S}^2$, B a homeomorph of $\mathbf{S}^3$ and $A \cap B = \{z_0\}$ is a single point.

(5) Suppose that maps between three spheres forms a fibration. Show that $k = n - 1$.

$$\begin{array}{ccc} \mathbf{S}^k & \xrightarrow{\iota} & \mathbf{S}^{n+k} \\ & & \downarrow p \\ & & \mathbf{S}^n \end{array}$$

By giving examples, show that such a fibration is indeed possible for some n.

(6) Find the homology, cohomology and homotopy groups

a. $H_*(\mathbb{R}\mathbf{P}^3, \mathbb{Z})$;

b. $H_*(\mathbb{R}\mathbf{P}^3, \mathbb{Z}/2\mathbb{Z})$;

c. $H^*(\mathbb{R}\mathbf{P}^3, \mathbb{Z})$;

d. $\pi_k(\mathbb{R}\mathbf{P}^3, x_0)$, for $k = 1, 2, 3$.

(7) Compute the deRham cohomology of $\mathbf{S}^2$ in *four* distinct ways. You may assume the singular cohomology groups $H^*(\mathbf{S}^2, \mathbb{R})$. (Hint: one way could be from first principles using the Poincaré Lemma.)

(8) The surface parameterized by $X : (u, v) \mapsto (\cos u, \sin u, 2\cos v, 2\sin v)$ is a torus in $\mathbb{R}^4$ with the induced metric $ds^2 = X^* ds^2_{\mathbb{R}^4}$. Show that this torus cannot be isometrically immersed into $\mathbb{R}^3$.

(9) Let M^2 be a smooth closed connected orientable surface with Gauss curvature $K(x) < 0$ and genus g. Let $\{\gamma_1, \cdots, \gamma_\ell\}$ be a system of pairwise nonintersecting closed geodesics of M. (A *closed geodesic* is a smoothly embedded loop $\gamma_i : \mathbf{S}^1 \to M$ whose geodesic curvature vanishes identically $\kappa_i \equiv 0$.) Show that $\ell \leq 3g - 3$. (Hint: can two nonintersecting closed geodesics be freely homotopic?)

(10) Let f be a smooth function on a smooth manifold M^n. A critical point $x_0 \in M$ is *nondegenerate* if in some coordinate system the Hessian matrix of second partials $\frac{\partial^2 f}{\partial x^i \partial x^j}(x_0)$ is nonsingular at x_0. A function all of whose critical points are nondegenenerate is called a *Morse function.* Show that the critical points of a Morse function are isolated. Given any smooth function $f \in C^\infty(M)$, show that for every $\varepsilon > 0$ there is a Morse function $g \in C^\infty(M)$ with $\sup_{x \in M} |f(x) - g(x)| < \varepsilon$.

(11) Let $M \subset \mathbb{R}^{n+1}$ be the graph $x^{n+1} = f(x^1, \dots, x^n)$ where $f \in C^\infty(\Omega)$ and $\Omega \subset \mathbb{R}^n$ is a convex open set. Suppose the Hessian of f has constant rank *i.e.* for some k such that $0 \leq k < n$ and every $x \in \Omega$, $rank\left(\frac{\partial^2 f(x)}{\partial x^i \partial x^j}\right) = k$. Show that M is foliated by $(n - k)$ dimensional manifolds whose tangent vectors lie in $\ker(dG)$ where $G : M \to \mathbb{R}^n$ is given by $G(x) = \left(\frac{\partial f}{\partial x^1}, \cdots, \frac{\partial f}{\partial x^n}\right)$. Show that a connected component L of such a "leaf" manifold locally coincides with points where G is constant. In fact, the graph of L is an affine subspace of $\mathbb{R}^{n+1}$ restricted to $\Omega \times \mathbb{R}$.