UNIVERSITY OF UTAH DEPARTMENT OF MATHEMATICS

Ph.D. Preliminary Examination in Differential Equations

January 4th, 2016.

Instructions: This examination has two parts consisting of five problems in part A and five in part B. You are to work three problems from part A and three problems from part B. If you work more than the required number of problems, then state which problems you wish to be graded, otherwise the first three will be graded.

In order to receive maximum credit, solutions to problems must be clearly and carefully presented and should be detailed as possible. All problems are worth 20 points.

A. Ordinary Differential Equations: Do three problems for full credit

A1. The simple pendulum consists of a point particle of mass m suspended from a fixed point by a massless rod of length L, which is allowed to swing in a vertical plane. If friction is ignored then the equation of motion is

$$\ddot{x} + \omega^2 \sin x = 0, \quad \omega^2 = \frac{g}{L},$$

where x is the angle of inclination of the rod with respect to the downward vertical and g is the gravitational constant.

(a) Using conservation of energy, show that the angular velocity of the pendulum satisfies

$$\dot{x} = \pm\sqrt{2}(C + \omega^2 \cos x)^{1/2},$$

where C is an arbitrary constant. Express C in terms of the total energy of the system.

(b) Sketch the phase diagram of the pendulum equation in the $(x, \dot{x})$-plane. Illustrate the one-parameter family of curves given by part (a) for different values of C. Take $-3\pi \leq x \leq 3\pi$. Indicate the fixed points of the system and the separatrices - curves linking the fixed points. Give a physical interpretation of the underlying trajectories in the two distinct dynamical regimes $|C| < \omega^2$ and $|C| > \omega^2$.

(c) Show that in the regime $|C| < \omega^2$, the period of oscillations is

$$T = 4\sqrt{\frac{L}{g}} K(\sin x_0/2),$$

where $\dot{x} = 0$ when $x = x_0$ and K is the complete elliptic integral of the first kind, which is defined by

$$K(\alpha) = \int_0^{\pi/2} \frac{1}{\sqrt{1 - \alpha^2 \sin^2 u}} du.$$

(d) For small amplitude oscillations, the pendulum equation can be approximated by the linear equation

$$\ddot{x} + \omega^2 x = 0.$$

Solve this equation for the initial conditions $x(0) = A$, $\dot{x}(0) = 0$ and sketch the phase-plane for different values of A. Compare with the phase-plane for the full nonlinear equation in part (b).

A2. Suppose $A(t)$ is a real $n \times n$ matrix function which is smooth in t and periodic of period $T > 0$. Consider the linear differential equation in $\mathbf{R}^n$

$$\begin{cases} \dfrac{dx}{dt} = A(t)\,x, \\ x(0) = x_0. \end{cases} \tag{1}$$

Let $\Phi(t)$ be the fundamental matrix solution with $\Phi(0) = I$.

(a) Define: *Floquet Matrix, Floquet Multiplier and Floquet Exponent.* How are these related to $\Phi(t)$? State the necessary and sufficient conditions so that (1) has a nonzero T-periodic solution.

(b) Prove that the zero solution is unstable for the system $\dot{x} = A(t)x$, where

$$A(t) = \begin{pmatrix} 1 & 1 \\ 0 & \dot{h}(t)/h(t) \end{pmatrix},$$

and $h(t) = 2 + \sin t - \cos t$.

A3. Consider Mathieu's equation for a parametric oscillator:

$$\ddot{x} + (\alpha + \beta \cos t)x = 0.$$

(a) Suppose that $\alpha \approx 1, \beta \approx 0$. Use a perturbation expansion in β to show that the transition curves for Mathieu's equation are given approximately by

$$\alpha = 1 - \frac{\beta^2}{12}, \quad \alpha = 1 + \frac{5}{12}\beta^2.$$

(b) Now suppose that $\alpha \approx 1/4 + \alpha_1\beta,\ \beta \approx 0$. In the unstable region near $\alpha = 1/4$, solutions of Mathieu's equation are of the form

$$c_1 e^{\sigma t} q_1(t) + c_1 e^{\sigma t} q_1(t)$$

where σ is real and positive, and q_1, q_2 are 4π–periodic. Derive the second order equation for q_1, q_2 and perform a power series expansion in β to show that $\sigma \approx \pm\beta\sqrt{1/4 - \alpha_1^2}$.

(c) Use part (b) to deduce that solutions of the damped Mathieu equation

$$\ddot{x} + \kappa\dot{x} + (\alpha + \beta \cos t)x = 0,$$

where $\kappa = \kappa_1\beta + \mathcal{O}(\beta^2)$, are stable if to first order in β,

$$\alpha < \frac{1}{4} - \frac{\beta}{2}\sqrt{1 - \kappa_1^2} \text{ or } \alpha > \frac{1}{4} + \frac{\beta}{2}\sqrt{1 - \kappa_1^2}.$$

A4. Consider a linear chain of $2N$ atoms consisting of two different masses m, M $M > m$, placed alternately. The atoms are equally spaced with lattice spacing a with nearest neighbor interactions represented by Hookean springs with spring constant β. Label the light atoms by even integers $2n, n = 0, ..., N-1$ and the heavy atoms by odd integers $2n-1, n = 1, ..., N$. Denoting their displacements from equilibrium by the variables U_{2n} and V_{2n-1} respectively, Newton's law of motion gives

$$m\ddot{U}_{2n} = \beta\,[V_{2n-1} + V_{2n+1} - 2U_{2n}]$$
$$M\ddot{V}_{2n-1} = \beta\,[U_{2n} + U_{2n-2} - 2V_{2n-1}]$$

Assume periodic boundary conditions $U_0 = U_{2N}$ and $V_1 = V_{2N+1}$.

(a) Sketch the configuration of atoms and briefly explain how the dynamical equations arise from Newton's law of motion.

(b) Assuming a solution of the form

$$U_{2n} = \Phi \mathrm{e}^{2inka} \mathrm{e}^{-i\omega t}, \quad V_{2n+1} = \Psi \mathrm{e}^{i(2n+1)ka} \mathrm{e}^{-i\omega t},$$

derive an eigenvalue equation for the amplitudes (Φ, Ψ) and determine the eigenvalues.

(c) Using part (b), show that there are two branches of solution and determine the speed w/k on the two branches for small k.

A5. (a) Give definitions for the following: *invariant set, attracting set, ω-limit set.*

(b) Determine the invariant sets and the attracting set of the dynamical system

$$\begin{aligned} \dot{x} &= -y + x(1 - x^2 - y^2) \\ \dot{y} &= x + y(1 - x^2 - y^2) \\ \dot{z} &= \alpha > 0. \end{aligned}$$

Also sketch the flow.

(c) Describe what happens to the flow if we identify the points $(x, y, 0)$ and $(x, y, 2\pi)$ in the planes $z = 0$ and $z = 2\pi$.

(d) By explicitly constructing solutions on the invariant torus $x^2 + y^2 = 1, 0 \leq z < 2\pi$, show that the torus is only an attractor if α is irrational.

B. Partial Differential Equations. Do three problems to get full credit

B1. Suppose that the eigenvalues λ_n and normalized eigenfunctions $\phi_n(\mathbf{x})$ of the Dirichlet problem for the Laplacian in a bounded domain D are known: that is,

$$\nabla^2\phi_n = \lambda_n\phi_n \text{ in } D, \quad \phi_n = 0 \text{ on } \partial D$$

for all integers $n \geq 0$

(a) Using Green's Theorem show that $\lambda_n < 0$.

(b) Derive the eigenfunction expansion of the Green's function $G(\mathbf{x}, \mathbf{y})$ for the Helmholtz equation

$$\nabla^2 G + k^2 G = \delta(\mathbf{x} - \mathbf{y}) \text{ in } D, \quad G = 0 \text{ on } \partial D,$$

assuming that $k^2 + \lambda_n \neq 0$ for all $n \geq 0$.

(c) Now suppose that $k^2+\lambda_n = 0$ for some n. Show how to construct a generalized Green's function by solving

$$(\nabla^2 - \lambda_n)G(\mathbf{x}, \mathbf{y}) = \delta(\mathbf{x} - \mathbf{y}) + c\phi_n(\mathbf{x})\phi_n(\mathbf{y})$$

via an eigenfunction expansion with a suitable choice of c.

B2. Consider the wave equation

$$u_{tt} - c^2 u_{xx} = 0, \qquad t > 0, x \in \mathbf{R}.$$

along with initial conditions

$$u(x, 0) = g(x), \qquad u_t(x, 0) = h(x).$$

(a) Assuming c is constant, derive d'Alembert's Formula

$$u(x,t) = \frac{1}{2}\left[g(x+ct) + g(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\xi)\, d\xi.$$

(b) Determine the solution for the initial data $u(x, 0) = 1$ if $|x| < a$, $u(x, 0) = 0$ if $x > |a|$; $u_t(x, 0) = 0$

(c) Determine the solution for the initial data $u(x, 0) = 0$; $u_t(x, 0) = 1$ if $|x| < a$, $u_t(x, 0) = 0$ if $x > |a|$

B3. Suppose that $\rho(x, t)$ is the number density of cars per unit length along a road, x being distance along the road, such that

$$\frac{\partial\rho}{\partial t} + \frac{\partial[\rho(1-\rho)]}{\partial x} = 0.$$

(a) Show that ρ is constant along the characteristics

$$\frac{dx}{dt} = 1 - 2\rho,$$

and derive the following Rankine-Hugoniot condition for the speed of a shock $x = S(t)$:

$$\frac{dS}{dt} = \frac{[\rho(1-\rho)]_-^+}{[\rho]_-^+}.$$

(b) A queue is building up at a traffic light $x = 1$ so that, when the light turns to green at $t = 0$,

$$\rho(x,0) = \begin{cases} 0, & \text{if } x < 0 \text{ and } x > 1; \\ x, & \text{if } 0 < x < 1. \end{cases}$$

Solve the corresponding characteristic equations, and sketch the resulting characteristic curves. Deduce that a collision first occurs at $x = 1/2$ when $t = 1/2$, and that thereafter there is a shock such that

$$\frac{dS}{dt} = \frac{S + t - 1}{2t}.$$

B4. Suppose that $u(\mathbf{x})$ is a $\mathcal{C}^2$ harmonic function in the domain $\Omega \subset \mathbf{R}^n$, so $\Delta u = 0$ in Ω.

(a) Prove the *mean value property:* if $\mathbf{x} \in \Omega$ and $r > 0$ is chosen such that $B_r(\mathbf{x}) \subset \Omega$ (ball of radius r centered at $\mathbf{x}$) then

$$u(\mathbf{x}) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B_r(\mathbf{x})} u(\mathbf{s}) d\mathbf{s},$$

where ω_n is the measure of ∂B_1. Hence show that

$$u(\mathbf{x}) \leq \frac{n}{\omega_n r^n} \int_{B_r(\mathbf{x})} u(\mathbf{y}) d\mathbf{y}.$$

(b) Assuming Ω is connected, prove that u can attain its maximum value at an interior point $x \in \Omega$, only if u is constant.

B5. Consider the equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = \epsilon \frac{\partial^2 u}{\partial x^2},$$

for $-\infty < x < \infty$ and $t > 0$.

(a) Look for a traveling wave solution $u(x,t) = U(z)$, $z = (x - Vt)/\epsilon$, with velocity $V > 0$ and $U(z) \to U_\pm$ as $z \to \pm\infty$. Solve the resulting ODE for $U(z)$ and deduce that

$$V = \frac{[U^2/2]_{-\infty}^{\infty}}{[U]_{-\infty}^{\infty}}$$

(b) Discuss how the traveling wave solution relates to shock solutions of the quasilinear equation obtained by setting $\epsilon = 0$.

(c) Using phase-plane analysis, show that the wave profile U is monotonically decreasing, that is, $dU/dz < 0$.