University of Utah, Department of Mathematics
May 2016, Algebra Qualifying Exam

Show all your work, and provide reasonable justification for your answers. You may attempt as many problems as you wish; five correct solutions count as a pass.

1. Let G be a group (not necessarily finite), and suppose that H is a subgroup of index n. Show that there is a normal subgroup N of G with $n! \geqslant [G:N] \geqslant n$.

2. Determine, up to isomorphism, the number of groups of order 70.

3. Let p be a prime integer, and G a group in which g^p is the identity for each g in G. Show that G must be abelian if $p = 2$. Give an example where G is not abelian.

4. Let $R = \mathbb{Q}[x]$, and let M be the cokernel of the map

$$R^2 \xrightarrow{\begin{pmatrix} x & 0 \\ x & x^2 \\ 1 & 1 \end{pmatrix}} R^3.$$

   Write M as a direct sum of cyclic R-modules.

5. Compute the characteristic polynomial, the minimal polynomial, and the Jordan form of the matrix

$$\begin{pmatrix} 3 & 1 & -1 \\ 2 & 2 & -1 \\ 2 & 2 & 0 \end{pmatrix}.$$

6. Let $K[x]$ be a polynomial ring over a field K, and let n be a positive integer. Classify, up to isomorphism, all finitely generated modules over the ring $K[x]/(x^n)$.

7. Factor $11x^5 - 11x^4 + 14x^2 - 21x + 7$ into irreducible polynomials in $\mathbb{Q}[x]$.

8. Prove that the polynomial $x^5 - x - 1$ has no root in $\mathbb{F}_9$, and that it is irreducible over $\mathbb{F}_3$. Determine the integers n for which $x^5 - x - 1$ is irreducible over $\mathbb{F}_{3^n}$.

9. Show that $K = \mathbb{Q}(\sqrt{1+\sqrt{3}})$ is not Galois over $\mathbb{Q}$, and compute $[K:\mathbb{Q}]$.

10. Let p be a prime integer, and set $f(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$. Suppose a prime integer q divides $f(a)$ for some integer a, prove that either $q = p$ or $q \equiv 1 \bmod p$.

    Use this to prove that the arithmetic sequence $1,\ 1+p,\ 1+2p,\ \ldots$ contains infinitely many prime integers.