2. Koszul modules

Suppose that V is an n-dimensional $\mathbf{k}$-vector space and fix a subspace $K \subseteq \bigwedge^2 V$ with $\dim(K) = m$. We denote by $S := \mathrm{Sym}(V)$ the symmetric algebra over V and consider the Koszul complex resolving the residue field $\mathbf{k}$:

$$\cdots \longrightarrow \bigwedge^3 V \otimes S \xrightarrow{\delta_3} \bigwedge^2 V \otimes S \xrightarrow{\delta_2} V \otimes S \xrightarrow{\delta_1} S.$$

Truncating this complex to the last three terms, and restricting δ_2 along the inclusion $\iota : K \hookrightarrow \bigwedge^2 V$ we obtain a 3-term complex

$$K \otimes S \xrightarrow{\quad \delta_2|_{K\otimes S} \quad} V \otimes S(1) \xrightarrow{\quad \delta_1 \quad} S(2). \tag{11}$$

The *Koszul module* associated to the pair (V, K) is the middle homology of the complex (11). We make the convention that K is placed in degree zero, so that $W(V, K)$ is a graded S-module generated in degree zero. In particular, the degree q component of $W(V, K)$ is given by

$$W_q(V, K) = \text{middle homology of } \left(K \otimes \mathrm{Sym}^q V \longrightarrow V \otimes \mathrm{Sym}^{q+1} V \longrightarrow \mathrm{Sym}^{q+2} V\right)$$

The formation of the Koszul module $W(V, K)$ is natural in the following sense. An inclusion $K \subseteq K'$ induces a surjective morphism of graded S-modules

$$W(V, K) \twoheadrightarrow W(V, K'), \tag{12}$$

that is, *bigger* subspaces $K \subseteq \bigwedge^2 V$ correspond to *smaller* Koszul modules. For instance, we have that $W(V, K) = 0$ if and only if $K = \bigwedge^2 V$. We'll be interested more generally in studying Koszul modules of finite length, that is, those that satisfy $W_q(V, K) = 0$ for $q \gg 0$. Since $W(V, K)$ is generated in degree zero, the vanishing $W_q(V, K) = 0$ for some $q \geq 0$ implies that $W_{q'}(V, K) = 0$ for all $q' \geq q$.

We write $\iota^\vee : \bigwedge^2 V^\vee \twoheadrightarrow K^\vee$ for the dual to the inclusion ι, let $K^\perp := \ker(\iota^\vee) \subseteq \bigwedge^2 V^\vee$ and define the *resonance variety* $\mathcal{R}(V, K)$ by

$$\mathcal{R}(V, K) := \left\{ a \in V^\vee : \text{ there exists } b \in V^\vee \text{ such that } a \wedge b \in K^\perp \setminus \{0\} \right\} \cup \{0\}. \tag{13}$$

Lemma 11. *The resonance variety $\mathcal{R}(V, K)$ coincides with the set-theoretic support of $W(V, K)$ in the affine space $V^\vee$.*

Proof. We let $\mathbf{P} = \mathbb{P}V^\vee$ denote the projective space of one dimensional subspaces of $V^\vee$, and consider the **Koszul sheaf** $\mathcal{W}(V, K)$, defined by considering the complex of sheaves associated to (11): $\mathcal{W}(V, K)$ is the middle homology of

$$K \otimes \mathcal{O}_{\mathbf{P}} \longrightarrow V \otimes \mathcal{O}_{\mathbf{P}}(1) \longrightarrow \mathcal{O}_{\mathbf{P}}(2)$$

Since $W(V,K)$ is a graded module, its set-theoretic support is the affine cone over the support of $\mathcal{W}(V,K)$.

If we write $\Omega = \Omega^1_{\mathbf{P}}$ for the sheaf of differential forms, then the Euler sequence

$$0 \longrightarrow \Omega \longrightarrow V \otimes \mathcal{O}_{\mathbf{P}}(-1) \longrightarrow \mathcal{O}_{\mathbf{P}} \longrightarrow 0 \tag{14}$$

yields the identification $\ker(V \otimes \mathcal{O}_{\mathbf{P}}(1) \longrightarrow \mathcal{O}_{\mathbf{P}}(2)) = \Omega(2)$, so that

$$\mathcal{W}(V,K) = \operatorname{coker}(K \otimes \mathcal{O}_{\mathbf{P}} \longrightarrow \Omega(2)).$$

It follows that the support of $\mathcal{W}(V,K)$ is the locus where the map $K \otimes \mathcal{O}_{\mathbf{P}} \longrightarrow \Omega(2)$ fails to be surjective. By Nakayama's lemma, this is a condition that can be checked on fibers.

Consider a point $p = [f] \in \mathbf{P}$, where $0 \neq f \in V^{\vee}$. The restriction of (14) to the fiber at p identifies with

$$0 \longrightarrow \ker(f) \longrightarrow V \xrightarrow{\ f\ } \mathbf{k} \longrightarrow 0\ , \tag{15}$$

so the restriction of the map $K \otimes \mathcal{O}_{\mathbf{P}} \longrightarrow \Omega(2)$ to the fiber at p is given by the contraction by f map $K \longrightarrow \ker(f)$. It follows that p is in the support of $\mathcal{W}(V,K)$ if and only if the corresponding sequence

$$K \longrightarrow V \xrightarrow{\ f\ } \mathbf{k}$$

fails to be exact in the middle. This is equivalent to the dual sequence

$$\mathbf{k} \xrightarrow{\cdot f} V^{\vee} \xrightarrow{\wedge f} K^{\vee}, \tag{16}$$

where the second map is the composition $V^{\vee} \xrightarrow{\wedge f} \bigwedge^2 V^{\vee} \twoheadrightarrow K^{\vee}$. It follows that a cycle in (16) is an element $g \in V^{\vee}$ with $g \wedge f \in K^{\perp}$, and g gives a non-trivial homology class if and only if g is not a multiple of f, that is, if $g \wedge f \neq 0$. Using (13) the existence of such g is equivalent to the fact that $f \in \mathcal{R}(V,K)$, which concludes our proof. ☐

It follows from Lemma 11 that $W(V,K)$ has finite length if and only $\mathcal{R}(V,K) = \{0\}$. In view of (13), this last condition is equivalent to the fact that the linear subspace $\mathbb{P}K^{\perp} \subseteq \mathbb{P}(\bigwedge^2 V^{\vee})$ is disjoint from the Grassmann variety

$$\mathbf{G} := \mathrm{Gr}_2(V^{\vee})$$

in its Plücker embedding, which can happen only when $m = \operatorname{codim}(\mathbb{P}K^{\perp}) > \dim(\mathbf{G}) = 2n-4$. Summarizing, we have the following equivalences:

$$\mathbb{P}K^{\perp} \cap \mathbf{G} = \emptyset \Longleftrightarrow \mathcal{R}(V,K) = \{0\} \Longleftrightarrow \dim_{\mathbf{k}} W(V,K) < \infty. \tag{17}$$

Moreover, if the equivalent statements in (17) hold, then $m \geq 2n-3$. The following theorem gives a sharp vanishing result for the graded components of a Koszul module with vanishing resonance.

Theorem 12. *Suppose that $n \geq 3$. If* $\mathrm{char}(\mathbf{k}) = 0$ *or* $\mathrm{char}(\mathbf{k}) \geq n-2$*, then we have the equivalence*

$$\mathcal{R}(V,K) = \{0\} \Longleftrightarrow W_q(V,K) = 0 \textit{ for } q \geq n-3. \tag{18}$$

Exercise 13. Check Theorem 12 in the case when $n = 3$.

Exercise 14. Show that if $\mathcal{R}(V,K) = \{0\}$ then there exists a subspace $K' \subseteq K$ with $\dim(K') = 2n-3$ such that $\mathcal{R}(V,K') = \{0\}$. Conclude that the implication "$\Longrightarrow$" in (18) reduces to the case when $\dim(K) = 2n-3$.

Proof of Theorem 12. The implication "$\Longleftarrow$" follows from (17). To prove "$\Longrightarrow$", we assume that (V,K) is such that $\mathcal{R}(V,K) = \{0\}$ and $\dim(K) = 2n-3$. With notation as in the proof of Lemma 11, it follows that the natural map $\alpha : K \otimes \mathcal{O}_{\mathbf{P}} \longrightarrow \Omega(2)$ is surjective, and therefore it gives rise to an exact Buchsbaum–Rim complex $\mathcal{B}_\bullet = \mathbf{BR}_\bullet(\alpha)$ with

$$\mathcal{B}_0 = \Omega(2),\ \mathcal{B}_1 = K \otimes \mathcal{O}_{\mathbf{P}},$$

$$\begin{aligned}\mathcal{B}_r &= \bigwedge^{n+r-2} K \otimes \det\left(\Omega^\vee(-2)\right) \otimes \mathrm{D}^{r-2}\left(\Omega^\vee(-2)\right)\\ &= \bigwedge^{n+r-2} K \otimes \mathcal{O}(-n-2r+6) \otimes \mathrm{D}^{r-2}(\Omega^\vee) \text{ for } r = 2,\cdots,n-1.\end{aligned}$$

The condition $W_{n-3}(V,K) = 0$ is equivalent to the fact that after twisting by $\mathcal{O}_{\mathbf{P}}(n-3)$, the induced map on global sections

$$H^0(\mathbf{P},\mathcal{B}_1(n-3)) \longrightarrow H^0(\mathbf{P},\mathcal{B}_0(n-3)) \tag{19}$$

is surjective. Since $\mathcal{B}_\bullet(n-3)$ is an exact complex, its hypercohomology groups are all zero. Using the hypercohomology spectral sequence, in order to prove the surjectivity of (19) it suffices to check that the sheaves $\mathcal{B}_r(n-3)$ have no cohomology (in fact, it is enough that $H^{r-1}(\mathbf{P},\mathcal{B}_r(n-3)) = 0$) for $r = 2,\cdots,n-1$. Since $0 \leq r-2 \leq n-3$, it follows from our hypothesis that $p = \mathrm{char}(\mathbf{k})$ satisfies $p = 0$ or $p > r-2$, thus $\mathrm{D}^{r-2}(\Omega^\vee) = \mathrm{Sym}^{r-2}(\Omega^\vee)$. It follows that

$$\mathcal{B}_r(n-3) = \bigwedge^{n+r-2} K \otimes \mathrm{Sym}^{r-2}\left(\Omega^\vee\right) \otimes \mathcal{O}(-2r+3), \text{ for } r = 2,\cdots,n-1,$$

and it suffices to check that $\mathrm{Sym}^{r-2}(\Omega^\vee) \otimes \mathcal{O}(-2r+3)$ has no non-zero cohomology for $r = 2,\cdots,n-1$. Dualizing the Euler sequence (14) and taking symmetric powers we obtain a short exact sequence

$$0 \longrightarrow \mathrm{Sym}^{r-3} V \otimes \mathcal{O}(-r) \longrightarrow \mathrm{Sym}^{r-2} V \otimes \mathcal{O}(-r+1) \longrightarrow \mathrm{Sym}^{r-2}(\Omega^\vee) \otimes \mathcal{O}(-2r+3) \longrightarrow 0.$$

It is then enough to check that $\mathcal{O}(-r)$ and $\mathcal{O}(-r+1)$ have no non-zero cohomology when $r = 2,\cdots,n-1$, which follows from the fact that $-n < -r, -r+1 < 0$ and cohomology of line bundles on projective space vanishes in this range. $\square$

Remark 15. If you know about Castelnuovo-Mumford regularity, then you can replace the spectral sequence argument in the proof above with the following (which I learned from Rob Lazarsfeld). Dualizing the Euler sequence (14) we get that $\Omega^\vee(-1)$ has a two-step resolution $0 \longrightarrow \mathcal{O}(-1) \longrightarrow V^\vee \otimes \mathcal{O}$. Since $\mathcal{O}$ is 0-regular and $\mathcal{O}(-1)$ is 1-regular, we conclude that $\Omega^\vee(-1)$ is 0-regular. A tensor product of copies of $\Omega^\vee(-1)$ will then also be 0-regular. Under our assumptions, $\mathrm{D}^{r-2}(\Omega^\vee) \otimes \mathcal{O}(-r+2) = \mathrm{D}^{r-2}(\Omega^\vee(-1))$ is a direct summand in a tensor product of $(r-2)$ copies of $\Omega^\vee(-1)$, so it is itself 0-regular. Since $\mathcal{O}(-i)$ is i-regular for all i, it follows that $\mathcal{B}_r(n-3)$ is $(r-1)$-regular for $r \geq 2$. If we let $\mathcal{J} = \ker(\mathcal{B}_1(n-3) \longrightarrow \mathcal{B}_0(n-3))$ it follows that $\mathcal{J}$ has a resolution $\mathcal{B}_{\bullet \geq 2}(n-3)$, where the i-th term $\mathcal{B}_{i+2}(n-3)$ is $(i+1)$-regular. This implies that $\mathcal{J}$ is 1-regular, so that $H^1(\mathbf{P}, \mathcal{J}) = 0$. From the long exact sequence

$$\cdots \longrightarrow H^0(\mathbf{P}, \mathcal{B}_1(n-3)) \longrightarrow H^0(\mathbf{P}, \mathcal{B}_0(n-3)) \longrightarrow H^1(\mathbf{P}, \mathcal{J}) \longrightarrow \cdots$$

it follows that the map (19) is surjective, as desired.

Experimental evidence suggests that $\mathrm{char}(\mathbf{k}) \geq n-2$ is the precise hypothesis necessary for (18) to hold. Similarly, the vanishing range $q \geq n-3$ is optimal, as shown by the following.

Theorem 16. *Suppose* $\mathrm{char}(\mathbf{k}) = 0$ *or* $\mathrm{char}(\mathbf{k}) \geq n-2$, *and fix a subspace* $K \subseteq \bigwedge^2 V$. *If* $\mathcal{R}(V,K) = \{0\}$, *then*

$$\dim W_q(V,K) \leq \binom{n+q-1}{q} \frac{(n-2)(n-q-3)}{q+2} \qquad \text{for } q = 0, \ldots, n-4.$$

Moreover, equality holds for all q *if* $\dim(K) = 2n-3$.

The table below records some of the values of $\dim W_q(V,K)$ in the case when equality holds in Theorem 16.

$q \setminus n$	4	5	6	7	8
0	1	3	6	10	15
1	–	5	16	35	64
2	–	–	21	70	162
3	–	–	–	84	288
4	–	–	–	–	330

(20)

Exercise 17. Find a formula for $\dim(K \otimes \mathrm{Sym}^q V)$ and $\dim(W_q(V,0))$, and check that

$$\dim(W_q(V,0)) - \dim(K \otimes \mathrm{Sym}^q V) = \binom{n+q-1}{q} \frac{(n-2)(n-q-3)}{q+2} \text{ if } \dim(K) = 2n-3.$$

Proof of Theorem 16. Using (12) and Exercise 14, we are reduced to the case $\dim(K) = 2n-3$. We have that $W_q(V,K)$ is the cokernel of the natural map

$$\beta_q : K \otimes \mathrm{Sym}^q V \longrightarrow W_q(V,0).$$

When $q = n-3$, the source and target have the same dimension. By Theorem 12, $W_{n-3}(V,K) = 0$, so β_{n-3} is an isomorphism, and in particular it is injective. Since $\beta = \bigoplus_q \beta_q : K \otimes S \longrightarrow W(V,0)$ is a map of S-modules, whose source is free, it follows that the injectivity of β_{n-3} implies that of β_q for all $q \leq n-3$. This shows that

$$\dim(W_q(V,K)) = \dim(W_0(V,K)) - \dim(K \otimes \operatorname{Sym}^q V) \text{ for } q = 0, \cdots, n-3,$$

and the desired formula follows from Exercise 17. □