GRAPHICAL DESIGNS WORKSHEET BRIDGES

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Part 1: Combinations

Definition 1. (1) A *linear combination* of $v_1, \ldots, v_n \in \mathbb{R}^\ell$ is a vector $\sum_{i=1}^n \lambda_i v_i$, $\lambda_i \in \mathbb{R}$. The *linear hull* of $v_1, \ldots, v_n$ is the set of all linear combinations of $v_1, \ldots, v_n$:

$$S = \operatorname{lin}(v_1, \ldots, v_n) = \left\{ \sum_{i=1}^n \lambda_i v_i \,:\, \lambda_i \in \mathbb{R} \right\}.$$

S is called a *subspace* of $\mathbb{R}^\ell$.

(2) An *affine linear combination* of $v_1, \ldots, v_n \in \mathbb{R}^\ell$ is a vector $\sum_{i=1}^n \lambda_i v_i$, $\lambda_i \in \mathbb{R}, \sum_{i=1}^n \lambda_i = 1$. The *affine hull* of $v_1, \ldots, v_n$ is the set of all affine linear combinations of $v_1, \ldots, v_n$:

$$A = \operatorname{aff}(v_1, \ldots, v_n) = \left\{ \sum_{i=1}^n \lambda_i v_i \,:\, \lambda_i \in \mathbb{R}, \sum_{i=1}^n \lambda_i = 1 \right\}.$$

A is called an *affine space* in $\mathbb{R}^\ell$.

(3) A *cone combination* of $v_1, \ldots, v_n \in \mathbb{R}^\ell$ is a vector $\sum_{i=1}^n \lambda_i v_i$, $\lambda_i \geq 0$. The *conical hull* of $v_1, \ldots, v_n$ is the set of all cone combinations of $v_1, \ldots, v_n$:

$$K = \operatorname{cone}(v_1, \ldots, v_n) = \left\{ \sum_{i=1}^n \lambda_i v_i \,:\, \lambda_i \geq 0 \right\}.$$

K is called a *cone* in $\mathbb{R}^\ell$.

(4) A *convex combination* of $v_1, \ldots, v_n \in \mathbb{R}^\ell$ is a vector $\sum_{i=1}^n \lambda_i v_i$, $\lambda_i \geq 0, \sum_{i=1}^n \lambda_i = 1$. The *convex hull* of $v_1, \ldots, v_n$ is the set of all convex combinations of $v_1, \ldots, v_n$:

$$P = \operatorname{conv}(v_1, \ldots, v_n) = \left\{ \sum_{i=1}^n \lambda_i v_i \,:\, \lambda_i \geq 0, \sum_{i=1}^n \lambda_i = 1 \right\}.$$

P is called a *polytope* in $\mathbb{R}^\ell$.

Exercise 1. Compute the linear, affine, conical and convex hulls of the configurations

$$\mathcal{A} := \left\{ \begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}1\\3\end{pmatrix}, \begin{pmatrix}1\\4\end{pmatrix} \right\} \quad \text{and} \quad \mathcal{B} = \left\{ \begin{pmatrix}0\\0\\1\end{pmatrix}, \begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}1\\-1\\1\end{pmatrix}, \begin{pmatrix}-1\\-1\\1\end{pmatrix}, \begin{pmatrix}-1\\1\\1\end{pmatrix} \right\}$$

Definition 1 gives the *generator representation* of the various objects defined there. Each one of them also has a *constraint/inequality representation.*

(1) The subspace $S = \operatorname{lin}(v_1, \ldots, v_n) = \{x \in \mathbb{R}^\ell \,:\, Cx = 0\}$ for some matrix C.

(2) The affine space $A = \operatorname{aff}(v_1, \ldots, v_n) = \{x \in \mathbb{R}^\ell \,:\, Cx = b\}$ for some matrtx C and vector b.

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(3) The cone $K = \text{cone}(v_1, \dots, v_n) = \{x \in \mathbb{R}^\ell : Cx \geq 0\}$ for some matrix C.

(4) The polytope $P = \text{conv}(v_1, \dots, v_n) = \{x \in \mathbb{R}^\ell : Cx \geq b\}$ for some matrix C and vector b.

Exercise 2. (1) Write the constraint representation of the cone $K = \text{cone}(\mathcal{A})$.

(2) Write the constraint representation of the polytope $P = \text{conv}(\mathcal{B})$.

A *polyhedron* is a set of the form $\{x \in \mathbb{R}^\ell : Cx \geq b\}$. Polytopes are bounded polyhedra. In particular, $K = \{x : Cx \geq 0\}$ is a special case of a polyhedron.

Part II: Polyhedral Cones

In general, a cone in $\mathbb{R}^\ell$ is any set that is closed under positive scalings; if $x \in K$ and $\lambda > 0$ then $\lambda x \in K$. For example the circular cone in $\mathbb{R}^3$

$$\{(x, y, z)^\top : x^2 + y^2 \leq z^2\}$$

with apex at the origin has this property. Notice that $K = \text{cone}(v_1, \dots, v_n)$ also has this property. However, unlike the circular cone, K is *finitely generated*. Notice also that if we were to write the circular cone as the set of cone combinations of a collection of vectors, then we would need infinitely many vectors (generators). Similarly if we were to write the circular cone in terms of linear inequalities, we would need infinitely many inequalities. Finitely generated cones are finitely constrained, and are called *polyhedral cones*.

Definition 2. The *polar* of a cone K is

$$K^\circ = \{y \in \mathbb{R}^\ell : y^\top x \geq 0 \ \forall \ x \in K\}.$$

In words, K° consists of all vectors that make nonnegative inner product with all elements in K.

Exercise 3. Draw and write down both representations of K° for $K = \text{cone}(\mathcal{A})$.

Exercise 4. (1) Prove that K° is a cone.

(2) Suppose $K = \text{cone}(v_1, \dots, v_n)$ and $M = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix} \in \mathbb{R}^{\ell \times n}$ is a matrix with columns $v_1, v_2, \dots, v_n$. Prove that

(a) $K = \{Mx : x \geq 0\}$.

(b) $K^\circ = \{y : y^\top M \geq 0\}$.

Theorem 3 (Farkas Lemma). *For $M \in \mathbb{R}^{\ell \times n}$ and $b \in \mathbb{R}^\ell$, exactly one of the following is true:*

(1) either $Mx = b, \ x \geq 0$ has a solution

(2) or there exists $y \in \mathbb{R}^\ell$ such that $y^\top M \geq 0$ and $y^\top b < 0$.

Exercise 5. Prove Farkas Lemma.
Hint: interpret the two statements in terms of a cone and its polar.

Farkas Lemma is the foundation of linear programming and is an example of a *theorem of alternatives.* There are many such theorems in mathematics. For example, what would be the theorem of alternatives for linear systems:

For $M \in \mathbb{R}^{\ell \times n}$ and $b \in \mathbb{R}^\ell$ either $Mx = b$ has a solution or .....??

Theorem 4 (Caratheodory's theorem). *Let $K = \text{conv}(v_1, \dots, v_n) \subseteq \mathbb{R}^\ell$ be a cone. Then for any $b \in K$ there exists a linearly independent subset $\mathcal{U}$ of $\{v_1, \dots, v_n\}$ such that $b \in \text{cone}(\mathcal{U})$. In particular, if $M = \begin{bmatrix} v_1 & v_1 & \cdots & v_n \end{bmatrix}$, then $Mx = b, x \geq 0$ has a solution x whose support has size at most ℓ.*

Exercise 6. Prove Caratheodory's theorem.

Part III: Designs and Eigenpolytopes

Let $G = ([n], E)$ be a connected, undirected regular graph with normalized adjacency matrix AD^{-1}. Recall that both A and D are $n \times n$ matrices:

- $A_{ij} = 1$ if $ij \in E$ and 0 otherwise.
- D is a diagonal matrix with $D_{ii} = \deg(i)$ for all $i \in [n]$.

Suppose AD^{-1} has m eigenspaces ordered as $\Lambda_1 = \operatorname{span}\{\mathbb{1}\} < \ldots < \Lambda_m$. A subset $W \subset [n]$ with nonnegative weights $a_w : w \in W$ is a k-design of G if

$$\sum_{w \in W} a_w \varphi(w) = \frac{1}{|V|} \sum_{v \in V} \varphi(v), \;\; \forall \varphi \in \Lambda_1, \ldots, \Lambda_k.$$

Exercise 7. (1) Prove that every subset of $V = [n]$ will average the first eigenvector $\mathbb{1} = \varphi_1$ of AD^{-1}.

(2) Prove that there is no proper subset of $V = [n]$ (with positive weights) that can average all eigenvectors of AD^{-1}.

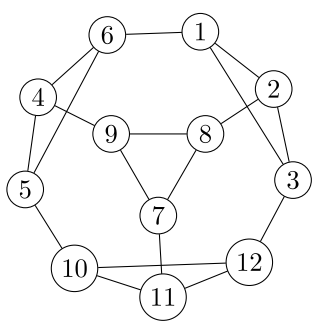
Let U denote a matrix whose rows are an eigenbasis of AD^{-1}, and let $I \subset [m]$ index a proper subset of eigenspaces of AD^{-1}.

(1) Let U_I denote the submatrix of U consisting of all rows corresponding to the eigenspaces Λ_i for $i \in I$.

(2) Let $\mathcal{U}_I$ denote the collection of columns of U_I, which we note may occur with repetition.

Definition 5 (C. Godsil, 1978)**.** The polytope $P_I = \operatorname{conv}(\mathcal{U}_I)$ is an *eigenpolytope* of G for the eigenvalues indexed by I.

Exercise 8. The Truncated Tetrahedral graph is shown below, along with an orthogonal basis of eigenvectors for AD^{-1}. The horizontal lines divide the eigenspaces.



$$\left[\begin{array}{cccccccccccc|c}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \lambda_1 \\
\hline
-1 & -1.5 & -0.5 & 1.5 & 2 & 1.5 & -0.5 & -1.5 & -1 & 1 & 0 & 0 & \\
2 & 1.5 & 1.5 & -0.5 & -1 & -1.5 & -1.5 & -0.5 & -1 & 0 & 1 & 0 & \lambda_2 \\
-1 & -0.5 & -1.5 & -1.5 & -1 & -0.5 & 1.5 & 1.5 & 2 & 0 & 0 & 1 & \\
\hline
0 & -1 & 1 & -1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & \\
-1 & 0 & 1 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 & 0 & \lambda_3 \\
0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & -1 & -1 & 0 & 1 & \\
\hline
-1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & \\
-1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 1 & 0 & \lambda_4 \\
-1 & 1 & 0 & 0 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & \\
\hline
1 & -1 & 0 & 0 & -1 & 1 & 1 & -1 & 0 & -1 & 1 & 0 & \lambda_5 \\
0 & -1 & 1 & 1 & -1 & 0 & 0 & -1 & 1 & -1 & 0 & 1 &
\end{array}\right.$$

In this exercise we will use eigenpolytopes to compute designs.

(1) Compute the eigenpolytopes P_3 and P_4.

(2) Compute the minimal designs of G that average $\Lambda_1, \Lambda_2, \Lambda_3, \Lambda_5$ using P_4.

(3) Compute the minimal designs of G that average $\Lambda_1, \Lambda_2, \Lambda_4, \Lambda_5$ using P_3.

Exercise 9. Let G be the graph of the 3-dimensional cube in $\mathbb{R}^3$. For this graph,

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}.$$

The matrix $AD^{-1} = \frac{1}{3}A$ has eigenvalues

$$1, -1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}$$

with eigenvectors:

$$\left(\begin{array}{cccccccc} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ \hline -1 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\ \hline 1 & 0 & 0 & -1 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 & 0 & 0 \\ \hline -1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & -1 & 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & 1 & 1 & 0 & 0 \end{array}\right)$$

Compute various types of designs of the cube.