TAUTOLOGICAL EQUATION IN $\overline{\mathcal{M}}_{3,1}$ VIA INVARIANCE THEOREM

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This paper is dedicated to Matteo S. Arcara,
who was born while the paper was at its last phase of preparation.

ABSTRACT. A new tautological equation of $\overline{\mathcal{M}}_{3,1}$ in codimension 3 is derived and proved, using the invariance condition explained in [1, 6, 8, 9]. An equivalent equation is independently found by Kimura–Liu [5] through a different method.

1. INTRODUCTION

This work is a continuation of [1]. We apply the same technique to $R^3(\overline{\mathcal{M}}_{3,1})$ to find a tautological equation. A general scheme and practical steps, as well as notations used in this paper, can be found in [8] and [1].

1.1. **Tautological rings.** One reference for tautological rings, which is close to the spirit of the present paper, is R. Vakil's survey article [10].

Let $\overline{\mathcal{M}}_{g,n}$ be the moduli stacks of stable curves. $\overline{\mathcal{M}}_{g,n}$ are proper, irreducible, smooth Deligne–Mumford stacks. The Chow rings $A^*(\overline{\mathcal{M}}_{g,n})$ over $\mathbb{Q}$ are isomorphic to the Chow rings of their coarse moduli spaces. The tautological rings $R^*(\overline{\mathcal{M}}_{g,n})$ are subrings of $A^*(\overline{\mathcal{M}}_{g,n})$, or subrings of $H^{2*}(\overline{\mathcal{M}}_{g,n})$ via cycle maps, generated by some "geometric classes" which will be described below.

The first type of geometric classes are the *boundary strata*. $\overline{\mathcal{M}}_{g,n}$ have natural stratification by topological types. The second type of geometric classes are the Chern classes of tautological vector bundles. These includes cotangent classes ψ_i, Hodge classes λ_k and κ-classes κ_l.

To give a precise definition of the tautological rings, some natural morphisms between moduli stacks of curves will be used. The *forgetful morphisms*

$$(1) \qquad \mathrm{ft}_i : \overline{\mathcal{M}}_{g,n+1} \to \overline{\mathcal{M}}_{g,n}$$

forget one of the $n + 1$ marked points. The *gluing morphisms*

$$(2) \qquad \overline{\mathcal{M}}_{g_1,n_1+1} \times \overline{\mathcal{M}}_{g_2,n_2+1} \to \overline{\mathcal{M}}_{g_1+g_2,n_1+n_2}, \quad \overline{\mathcal{M}}_{g-1,n+2} \to \overline{\mathcal{M}}_{g,n},$$

glue two marked points to form a curve with a new node. Note that the boundary strata are the images (of the repeated applications) of the gluing morphisms, up to factors in $\mathbb{Q}$ due to automorphisms.

Definition 1. The system of tautological rings $\{R^*(\overline{\mathcal{M}}_{g,n})\}_{g,n}$ is the smallest system of $\mathbb{Q}$-unital subalgebra (containing classes of type one and two, and is) closed under push-forwards via the forgetful and gluing morphisms.

The study of the tautological rings is one of the central problems in moduli of curves. The readers are referred to [10] and references therein for many examples and motivation. Note that the tautological rings are defined by generators and relations. Since the generators are explicitly given, *the study of tautological rings is the study of relations of tautological classes.*

1.2. **Invariance Constraints.** Here some ingredients in [8] and [9] will be briefly reviewed.

The moduli of curves can be stratified by topological types. Each boundary stratum can be conveniently presented by the (dual) graphs of its generic curves in the following way. To each stable curve C with marked points, one can associate a dual graph Γ. Vertices of Γ correspond to irreducible components. They are labeled by their geometric genus. Assign an edge joining two vertices each time the two components intersect. To each marked point, one draws an half-edge incident to the vertex, with the same label as the point. Now, the stratum corresponding to Γ is the closure of the subset of all stable curves in $\overline{\mathcal{M}}_{g,n}$ which have the same topological type as C. For each dual graph Γ, one can decorate the graph by assigning a monomial, or more generally a polynomial, of ψ to each half-edge and κ classes to each vertex. The tautological classes in $R^k(\overline{\mathcal{M}}_{g,n})$ can be represented by $\mathbb{Q}$-linear combinations of *decorated graphs. Since there is no κ, λ-classes involved in this paper, they will be left out of discussions below.*

For typesetting reasons, it is more convenient to denote a decorated graph by another notation, inspired by Gromov–Witten theory, called *gwi*. Given a decorated graph Γ.

- For the vertices of Γ of genus $g_1, g_2, \ldots$, assign a product of "brackets" $\langle \rangle_{g_1} \langle \rangle_{g_2} \ldots$. To simplify the notations, $\langle \rangle := \langle \rangle_0$.
- Assign each half-edge a symbol ∂^*. The external half-edges use super-indices $\partial^x, \partial^y, \ldots$, corresponding to their labeling. For

each pair of half-edges coming from one and the same edge, the same super-index will by used, denoted by Greek letters $(\mu, \nu, \ldots.)$ Otherwise, all half-edges should use different super-indices.

- For each decoration to a half-edge a by ψ-classes ψ^k, assign a subindex to the corresponding half-edge ∂_k^a.
- For each a given vertex $\langle\rangle_g$ with m half-edges, n external half-edges, an insertion is placed in the vertex $\langle \partial_{k_1}^x \partial_{k_2}^y \ldots \partial_{k'}^\mu \partial_{k''}^\nu \ldots \rangle_g$.

The key tool employed in this paper is the *existence* of linear operators

$$(3) \qquad \mathfrak{r}_l : R^k(\overline{\mathcal{M}}_{g,n}) \to R^{k-l+1}(\overline{\mathcal{M}}_{g-1,n+2}^\bullet), \quad l = 1, 2, \ldots,$$

where the symbol $\bullet$ denotes the moduli of possibly disconnected curves. $\mathfrak{r}_l$ is defined as an operation on the decorated graphs. The output graphs have two more markings, which are denoted by i, j. In terms of gwis,

$$(4)$$

$$\mathfrak{r}_l \left(\langle \partial_{k'}^\mu \ldots \rangle_{g'} \ldots \langle \partial_{k''}^\mu \ldots \rangle_{g''} \right)$$

$$= \frac{1}{2} \left(\langle \partial_{k'+l}^i \ldots \rangle_{g'} \ldots \langle \partial_{k''}^j \ldots \rangle_{g''} + \langle \partial_{k'}^j \ldots \rangle_{g'} \langle \partial_{k''+l}^i \ldots \rangle_{g''} \right)$$

$$+ \frac{1}{2} (-1)^{l-1} \left(\langle \partial_{k'+l}^j \ldots \rangle_{g'} \ldots \langle \partial_{k''}^i \ldots \rangle_{g''} + \langle \partial_{k'}^i \ldots \rangle_{g'} \langle \partial_{k''+l}^j \ldots \rangle_{g''} \right) + \ldots$$

$$+ \frac{1}{2} \sum_{m=0}^{l-1} (-1)^{m+1} \langle \partial_{l-1-m}^i \partial_m^j \partial_{k'}^\mu \ldots \rangle_{g'-1} \ldots \langle \partial_{k''}^\mu \ldots \rangle_{g''} + \ldots$$

$$+ \frac{1}{2} \sum_{m=0}^{l-1} (-1)^{m+1} \langle \partial_{k'}^\mu \ldots \rangle_{g'} \ldots \langle \partial_{l-1-m}^i \partial_m^j \partial_{k''}^\mu \ldots \rangle_{g''-1}$$

$$+ \frac{1}{2} \left(\sum_{m=0}^{l-1} (-1)^{m+1} \sum_{g=0}^{g'} \partial_{k'}^\mu \ldots \left(\langle \partial_{l-1-m}^i \rangle_g \langle \partial_m^j \rangle_{g'-g} \right) \right) \langle \partial_{k''}^\mu \ldots \rangle_{g''} + \ldots$$

$$+ \frac{1}{2} \langle \partial_{k'}^\mu \ldots \rangle_{g'} \left(\sum_{m=0}^{l-1} (-1)^{m+1} \sum_{g=0}^{g''} \partial_{k''}^\mu \ldots \left(\langle \partial_{l-1-m}^i \rangle_{g''} \langle \partial_m^j \rangle_{g''-g} \right) \right),$$

where the notation $\partial_k^\mu \ldots \left(\langle \partial_{l-1-m}^i \rangle_{g_1} \langle \partial_m^j \rangle_{g_2} \right)$ means that the half-edge insertions $\partial_k^\mu \ldots$ acts on the product of vertices $\langle \partial_{l-1-m}^i \rangle_{g_1} \langle \partial_m^j \rangle_{g_2}$ by Leibniz rule. Note that $\langle \ldots \rangle_{-1} := 0$.

Note that one class in $R^k(\overline{\mathcal{M}}_{g,n})$ may have more than one graphical presentations coming from tautological equations. It is highly non-trivial that certain combination of these graphical operations would

descend to operations on $R^k(\overline{\mathcal{M}}_{g,n})$. This was originally Invariance Conjecture 1 in [8], and now the following Invariance Theorem.

Theorem. ([9] Theorem 5) $\mathfrak{r}_l$ *is well-defined. That is, if*

$$E = 0$$

is a tautological equation in $R^k(\overline{\mathcal{M}}_{g,n})$,

$$\mathfrak{r}_l(E) = 0.$$

Remarks. (i) In terms of graphical operations, the first two lines of equation (4) stand for "cutting edges"; the middle two for "genus reduction"; the last two for "splitting vertices". These are explained in [8].

(ii) In this paper, only $l = 1$ case will be used.

1.3. The algorithm of finding tautological equations. Our way of finding this equation is fairly simple.

(a) By Graber–Vakil's $(*)$ Theorem [4] or Getzler–Looijenga's Hodge number calculations [3], there is a new equation in $R^3(\overline{\mathcal{M}}_{3,1})$.

(b) Apply Invariance Theorem (Theorem 5 [8]) to obtain the coefficients of the equation.

(b) gives a *necessary* condition. Combined with (a), this generates and proves the new equation.

In the case of $R^3(\overline{\mathcal{M}}_{3,1})$, we first identify 30 "potentially independent" decorated graphs with decorations coming from ψ-classes only. This is done in Section 2. A general combination of these 30 decorated graphs is written as

$$E = \sum_{k=1}^{30} c_k(k)$$

where (k) denote the k-th decorated graph and c_k are the unknown coefficient to be found. Suppose that

$$E = 0$$

is a tautological equation. Application of Invariance Theorem implies that

$$\mathfrak{r}_1(E) = 0,$$

where

$$\mathfrak{r}_1 : R^3(\overline{\mathcal{M}}_{3,1}) \to R^3(\overline{\mathcal{M}}_{2,3}^\bullet)$$

By analyzing the properties of the image in $R^3(\overline{\mathcal{M}}_{2,3}^\bullet)$, which is known by works in genus one and two, [1] we obtain a system of homogeneous

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[1]See [1] and references therein.

linear equations on c_i, which will be equations (5)-(53). This system is potentially very over-determined, as there are more equations than variables. However, the Invariance Theorem predicts that this system of linear equations uniquely determines c_k if there is a nontrivial tautological equation. ($c_k = 0$ is a trivial solution.)

1.4. **Motivation.** Our motivation was quite simple. In the earlier work [3, 1], the conjectural framework is shown to be valid in genus one and two. While it is satisfactory to learn that all previous equations can be derived in this framework, the Conjectures predicts the possibility of finding all tautological equations under this framework. This work is set to be the first step towards this goal.

The choice of codimension 3 in $\overline{\mathcal{M}}_{3,1}$ is almost obvious. First of all, the Invariance Conjectures works inductively. Given what we know about genus one and two, it is only reasonable to proceed to either $\overline{\mathcal{M}}_{2,n}$ for $n \geq 4$ or $\overline{\mathcal{M}}_{3,1}$. Secondly, one also knows from the Theorem of Graber–Vakil [4] that ψ^3 on $\overline{\mathcal{M}}_{3,1}$ is rational equivalent to a sum of boundary strata containing at least one (geometrical) genus zero component. Thirdly, Getzler and Looijenga [3] have shown that there is only one relation in codimension 3 in $\overline{\mathcal{M}}_{3,1}$. That makes it a reasonable place to start.

1.5. **Main result.** The main result of this paper is the following:

Theorem. *There is a new tautological equation for codimension 3 strata in* $\overline{\mathcal{M}}_{3,1}$.

$$
\begin{aligned}
\langle \partial_3^x \rangle_3 = & \frac{5}{72} \langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^{\mu_1} \partial^{\mu_3} \partial^{\mu_2} \rangle \langle \partial^{\mu_3} \rangle_2 + \frac{1}{252} \langle \partial^{\mu_1} \partial^{\mu_2} \partial^{\mu_2} \rangle \langle \partial_1^x \partial^{\mu_1} \rangle_2 \\
& + \frac{5}{72} \langle \partial^x \partial^{\mu_1} \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial_1^{\mu_2} \rangle_2 + \frac{5}{42} \langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial_1^{\mu_1} \partial^{\mu_2} \rangle_2 \\
& + \frac{41}{21} \langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^{\mu_1} \rangle_1 \langle \partial_1^{\mu_2} \rangle_2 + \frac{11}{40320} \langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \partial^{\mu_2} \rangle \langle \partial^{\mu_1} \partial^{\mu_3} \partial^{\mu_3} \rangle_1 \\
& + \frac{1}{13440} \langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \partial^{\mu_2} \rangle_1 \langle \partial^{\mu_1} \partial^{\mu_3} \partial^{\mu_3} \rangle + \frac{1}{8064} \langle \partial^x \partial^{\mu_1} \rangle_1 \langle \partial^{\mu_1} \partial^{\mu_2} \partial^{\mu_2} \partial^{\mu_3} \partial^{\mu_3} \rangle \\
& + \frac{191}{120960} \langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \partial^{\mu_2} \partial^{\mu_3} \partial^{\mu_3} \rangle \langle \partial^{\mu_1} \rangle_1 + \frac{1}{5040} \langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^{\mu_1} \partial^{\mu_2} \partial^{\mu_3} \partial^{\mu_3} \rangle_1 \\
& + \frac{1}{4032} \langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \rangle_1 \langle \partial^{\mu_1} \partial^{\mu_2} \partial^{\mu_3} \partial^{\mu_3} \rangle + \frac{17}{2880} \langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \partial^{\mu_3} \partial^{\mu_3} \rangle \langle \partial^{\mu_1} \partial^{\mu_2} \rangle_1 \\
& + \frac{1}{840} \langle \partial^x \partial^{\mu_1} \rangle_1 \langle \partial^{\mu_1} \partial^{\mu_2} \rangle_1 \langle \partial^{\mu_2} \partial^{\mu_3} \partial^{\mu_3} \rangle \\
& + \frac{1}{336} \langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \rangle_1 \langle \partial^{\mu_1} \rangle_1 \langle \partial^{\mu_2} \partial^{\mu_3} \partial^{\mu_3} \rangle \\
& + \frac{1}{126} \langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^{\mu_1} \rangle_1 \langle \partial^{\mu_2} \partial^{\mu_3} \partial^{\mu_3} \rangle_1 \\
& + \frac{23}{5040} \langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \partial^{\mu_2} \rangle \langle \partial^{\mu_1} \partial^{\mu_3} \rangle_1 \langle \partial^{\mu_3} \rangle_1 \\
& + \frac{17}{5040} \langle \partial^x \partial^{\mu_1} \rangle_1 \langle \partial^{\mu_1} \partial^{\mu_2} \partial^{\mu_3} \partial^{\mu_3} \rangle \langle \partial^{\mu_2} \rangle_1 \\
& + \frac{113}{2520} \langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \partial^{\mu_3} \partial^{\mu_3} \rangle \langle \partial^{\mu_1} \rangle_1 \langle \partial^{\mu_2} \rangle_1 \\
& + \frac{1}{210} \langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \partial^{\mu_3} \rangle \langle \partial^{\mu_1} \partial^{\mu_2} \partial^{\mu_3} \rangle_1 \\
& + \frac{1}{84} \langle \partial^x \partial^{\mu_1} \rangle_1 \langle \partial^{\mu_1} \partial^{\mu_2} \partial^{\mu_3} \rangle \langle \partial^{\mu_2} \partial^{\mu_3} \rangle_1 \\
& + \frac{211}{1260} \langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \partial^{\mu_3} \rangle \langle \partial^{\mu_1} \partial^{\mu_2} \rangle_1 \langle \partial^{\mu_3} \rangle_1 \\
& + \frac{1}{1260} \langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \rangle_1 \langle \partial^{\mu_1} \partial^{\mu_2} \partial^{\mu_3} \rangle \langle \partial^{\mu_3} \rangle_1 \\
& + \frac{1}{630} \langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^{\mu_1} \partial^{\mu_2} \partial^{\mu_3} \rangle_1 \langle \partial^{\mu_3} \rangle_1 \\
& + \frac{11}{140} \langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^{\mu_1} \partial^{\mu_3} \rangle_1 \langle \partial^{\mu_2} \partial^{\mu_3} \rangle_1 \\
& + \frac{4}{35} \langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \rangle \langle \partial^{\mu_1} \rangle_1 \langle \partial^{\mu_2} \partial^{\mu_3} \rangle_1 \langle \partial^{\mu_3} \rangle_1 \\
& + \frac{2}{105} \langle \partial^x \partial^{\mu_1} \rangle_1 \langle \partial^{\mu_1} \partial^{\mu_2} \partial^{\mu_3} \rangle \langle \partial^{\mu_2} \rangle_1 \langle \partial^{\mu_3} \rangle_1 \\
& + \frac{89}{210} \langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \partial^{\mu_3} \rangle \langle \partial^{\mu_1} \rangle_1 \langle \partial^{\mu_2} \rangle_1 \langle \partial^{\mu_3} \rangle_1 \\
& + \frac{1}{53760} \langle \partial^x \partial^{\mu_1} \partial^{\mu_1} \partial^{\mu_2} \partial^{\mu_2} \partial^{\mu_3} \partial^{\mu_3} \rangle + 0 \langle \partial^x \partial^{\mu_1} \partial^{\mu_2} \partial^{\mu_3} \rangle_1 \langle \partial^{\mu_1} \partial^{\mu_2} \partial^{\mu_3} \rangle.
\end{aligned}
$$

Remarks. (i) While our paper was under preparation, a preprint by
T. Kimura and X. Liu [5] appeared on the arxiv. There are two major
differences between our results. First, their choice of basis of codimen-
sion 3 strata in $\overline{\mathcal{M}}_{3,1}$ is different. They use $(3') := \langle \partial^{\mu_1} \partial^{\mu_2} \partial^{\mu_2} \rangle \langle \partial^x \partial_1^{\mu_1} \rangle_2$
instead of (3) below. We have checked that their equation is equivalent
to ours. Second, their approach is "traditional": knowing there must
be a relation from Graber–Vakil, they can then proceed to find the
coefficients based on the evaluation of the Gromov–Witten invariants
of $\mathbb{P}^1$.

Our approach is quite different. There are no computer-aided calcu-
lation of the Gromov–Witten invariants. Only linear algebra is involved
in the calculation.

(ii) This technique has been applied to prove some Faber type result
in tautological rings [2]. A corollary of [2] is that there is no relation
between ψ_1 and κ_1 in $R^1(\overline{\mathcal{M}}_{3,1})$. It is easy to see, however, that there
is a relation of the monomials in κ-classes and ψ-classes. On the other
hand, the reader may amuse himself with the following result.

Proposition 1. *There is no (new) relation among the classes in $R^3(\partial \overline{\mathcal{M}}_{3,1})$*
and ψ_1^2 in $R^2(\overline{\mathcal{M}}_{3,1})$.

This can be shown, for example, by the same technique used in
this paper. Let (k) denotes a basis of these strata. When one sets a
hypothetical equation $c_k(k) = 0$ and imposes the invariance condition,
the only solution is $c_k = 0$ for all k.

2. STRATA OF $\overline{\mathcal{M}}_{3,1}$

We start with enumerating codimension 3 strata in $\overline{\mathcal{M}}_{3,1}$.

Out of the several strata (allowing ψ classes) of codimension 3 in
$\overline{\mathcal{M}}_{3,1}$, many of them can be written in terms of the others using WDVV,
TRR's, Mumford–Getzler's, Getzler's and Belorousski-Pandharipande
equations. After applying those equations, we can write all of the strata
in terms of the following ones:

(1) $\qquad$ $\langle \partial_3^x \rangle_3$ $\qquad$ (2) $\quad$ $\langle \partial^x \partial^\alpha \partial^\beta \rangle \langle \partial^\mu \partial^\alpha \partial^\beta \rangle \langle \partial^\mu \rangle_2$

(3) $\qquad$ $\langle \partial^\mu \partial^\alpha \partial^\alpha \rangle \langle \partial_1^x \partial^\mu \rangle_2$ $\qquad$ (4) $\quad$ $\langle \partial^x \partial^\mu \partial^\mu \partial^\nu \rangle \langle \partial_1^\nu \rangle_2$

(5) $\qquad$ $\langle \partial^x \partial^\mu \partial^\nu \rangle \langle \partial_1^\mu \partial^\nu \rangle_2$ $\qquad$ (6) $\quad$ $\langle \partial^x \partial^\mu \partial^\nu \rangle \langle \partial^\mu \rangle_1 \langle \partial_1^\nu \rangle_2$

(7) $\qquad$ $\langle \partial^x \partial^\mu \partial^\nu \partial^\nu \rangle \langle \partial^\mu \partial^\alpha \partial^\alpha \rangle_1$ $\qquad$ (8) $\quad$ $\langle \partial^x \partial^\mu \partial^\nu \partial^\nu \rangle_1 \langle \partial^\mu \partial^\alpha \partial^\alpha \rangle$

(9) $\qquad$ $\langle \partial^x \partial^\mu \rangle_1 \langle \partial^\mu \partial^\nu \partial^\nu \partial^\alpha \partial^\alpha \rangle$ $\qquad$ (10) $\quad$ $\langle \partial^x \partial^\mu \partial^\nu \partial^\nu \partial^\alpha \partial^\alpha \rangle \langle \partial^\mu \rangle_1$

(11) $\qquad$ $\langle \partial^x \partial^\mu \partial^\nu \rangle \langle \partial^\mu \partial^\nu \partial^\alpha \partial^\alpha \rangle_1$ $\qquad$ (12) $\quad$ $\langle \partial^x \partial^\mu \partial^\nu \rangle_1 \langle \partial^\mu \partial^\nu \partial^\alpha \partial^\alpha \rangle$

(13) $\qquad$ $\langle \partial^x \partial^\mu \partial^\nu \partial^\alpha \partial^\alpha \rangle \langle \partial^\mu \partial^\nu \rangle_1$ $\qquad$ (14) $\quad$ $\langle \partial^x \partial^\mu \rangle_1 \langle \partial^\mu \partial^\nu \rangle_1 \langle \partial^\nu \partial^\alpha \partial^\alpha \rangle$

(15) $\qquad$ $\langle \partial^x \partial^\mu \partial^\nu \rangle_1 \langle \partial^\mu \rangle_1 \langle \partial^\nu \partial^\alpha \partial^\alpha \rangle$ $\qquad$ (16) $\quad$ $\langle \partial^x \partial^\mu \partial^\nu \rangle \langle \partial^\mu \rangle_1 \langle \partial^\nu \partial^\alpha \partial^\alpha \rangle_1$

(17) $\qquad$ $\langle \partial^x \partial^\mu \partial^\nu \partial^\nu \rangle \langle \partial^\mu \partial^\alpha \rangle_1 \langle \partial^\alpha \rangle_1$ $\qquad$ (18) $\quad$ $\langle \partial^x \partial^\mu \rangle_1 \langle \partial^\mu \partial^\nu \partial^\alpha \partial^\alpha \rangle \langle \partial^\nu \rangle_1$

(19) $\qquad$ $\langle \partial^x \partial^\mu \partial^\nu \partial^\alpha \partial^\alpha \rangle \langle \partial^\mu \rangle_1 \langle \partial^\nu \rangle_1$ $\qquad$ (20) $\quad$ $\langle \partial^x \partial^\mu \partial^\nu \partial^\alpha \rangle \langle \partial^\mu \partial^\nu \partial^\alpha \rangle_1$

(21) $\qquad$ $\langle \partial^x \partial^\mu \partial^\nu \partial^\alpha \rangle_1 \langle \partial^\mu \partial^\nu \partial^\alpha \rangle$ $\qquad$ (22) $\quad$ $\langle \partial^x \partial^\mu \rangle_1 \langle \partial^\mu \partial^\nu \partial^\alpha \rangle \langle \partial^\nu \partial^\alpha \rangle_1$

(23) $\qquad$ $\langle \partial^x \partial^\mu \partial^\nu \partial^\alpha \rangle \langle \partial^\mu \partial^\nu \rangle_1 \langle \partial^\alpha \rangle_1$ $\qquad$ (24) $\quad$ $\langle \partial^x \partial^\mu \partial^\nu \rangle_1 \langle \partial^\mu \partial^\nu \partial^\alpha \rangle \langle \partial^\alpha \rangle_1$

(25) $\qquad$ $\langle \partial^x \partial^\mu \partial^\nu \rangle \langle \partial^\mu \partial^\nu \partial^\alpha \rangle_1 \langle \partial^\alpha \rangle_1$ $\qquad$ (26) $\quad$ $\langle \partial^x \partial^\mu \partial^\nu \rangle \langle \partial^\mu \partial^\alpha \rangle_1 \langle \partial^\nu \partial^\alpha \rangle_1$

(27) $\langle \partial^x \partial^\mu \partial^\nu \rangle \langle \partial^\mu \rangle_1 \langle \partial^\nu \partial^\alpha \rangle_1 \langle \partial^\alpha \rangle_1$ $\quad$ (28) $\langle \partial^x \partial^\mu \rangle_1 \langle \partial^\mu \partial^\nu \partial^\alpha \rangle \langle \partial^\nu \rangle_1 \langle \partial^\alpha \rangle_1$

(29) $\langle \partial^x \partial^\mu \partial^\nu \partial^\alpha \rangle \langle \partial^\mu \rangle_1 \langle \partial^\nu \rangle_1 \langle \partial^\alpha \rangle_1$ $\quad$ (30) $\qquad$ $\langle \partial^x \partial^\mu \partial^\mu \partial^\nu \partial^\nu \partial^\alpha \partial^\alpha \rangle$

3. Setting $\mathfrak{r}_1(E) = 0$

Let E be a generic linear combination of these strata

$$E := \sum_{k=1}^{30} c_k(k),$$

where c_k are variables with values in $\mathbb{Q}$. The Invariance Conjectures predict

$$\mathfrak{r}_1(E) = 0.$$

For the output graphs of $\mathfrak{r}_1(E)$, we will pick a basis for the tautological algebra and set the coefficients of the basis equal to 0. This will produce a system of linear equations on c_k, (1)-(49), which then determines c_k completely. The final equation $E = 0$, with the specified coefficients, is thus obtained.

Note that the (new) half-edges i, j in the output graphs are always assumed to be symmetrized. Some of the graphs will be disconnected. They are easier to deal with as they involve less relations (e.g. WDVV). So let us start with the disconnected terms. In each equation, the first column is a basis vector, followed by its coefficient which is set to zero.

(5) $\qquad$ $\langle \partial^x \partial^\alpha \partial^\beta \rangle \langle \partial^j \partial^\alpha \partial^\beta \rangle \langle \partial_1^i \rangle_2 : \quad c_2 + c_4 = 0.$

(6) $\qquad$ $\langle \partial^x \partial^i \partial^\alpha \rangle \langle \partial^j \partial^\beta \partial^\beta \rangle \langle \partial_1^\alpha \rangle_2 : \quad 3c_3 - c_4 + \dfrac{1}{24} c_6 = 0.$

(7) $\langle\partial^j\partial^\nu\partial^\nu\rangle\langle\partial^\alpha\partial^\alpha\partial^\beta\rangle\langle\partial^x\partial^i\partial^\beta\rangle_1:$ $-\dfrac{1}{80}c_3 - c_8 + \dfrac{1}{24}c_{15} = 0.$

(8) $\langle\partial^i\partial^\nu\partial^\beta\rangle\langle\partial^\beta\partial^x\partial^\nu\rangle\langle\partial^j\partial^\alpha\partial^\alpha\rangle_1:$ $c_7 - c_8 - c_{11} = 0.$

(9) $\langle\partial^j\partial^\nu\partial^\nu\rangle\langle\partial^x\partial^\alpha\partial^\beta\rangle\langle\partial^i\partial^\alpha\partial^\beta\rangle_1:$ $\dfrac{1}{30}c_3 - c_{11} + \dfrac{1}{24}c_{25} = 0.$

(10) $\langle\partial^j\partial^\nu\partial^\nu\rangle\langle\partial^i\partial^\alpha\partial^\beta\rangle\langle\partial^x\partial^\alpha\partial^\beta\rangle_1:$ $\dfrac{1}{30}c_3 + 2c_8 - c_{12} + \dfrac{1}{24}c_{24} = 0.$

(11) $\langle\partial^j\partial^\nu\partial^\nu\rangle\langle\partial^x\partial^i\partial^\alpha\rangle\langle\partial^\alpha\partial^\beta\partial^\beta\rangle_1:$ $-\dfrac{1}{30}c_3 - c_7 + c_8 + \dfrac{1}{24}c_{16} = 0.$

(12) $\langle\partial^\mu\partial^\nu\partial^\nu\rangle\langle\partial^j\partial^\mu\partial^\alpha\rangle\langle\partial^i\partial^x\rangle_1\langle\partial^\alpha\rangle_1:$ $c_{14} - c_{15} + c_{18} - c_{24} = 0;$

(13) $\langle\partial^\nu\partial^\alpha\partial^\alpha\rangle\langle\partial^i\partial^x\partial^\nu\rangle\langle\partial^j\partial^\mu\rangle_1\langle\partial^\mu\rangle_1:$ $c_{15} - c_{17} + c_{25} = 0.$

(14) $\langle\partial^j\partial^\nu\partial^\nu\rangle\langle\partial^x\partial^\alpha\partial^\beta\rangle\langle\partial^i\partial^\alpha\rangle_1\langle\partial^\beta\rangle_1:$ $\dfrac{4}{5}c_3 - c_{16} + \dfrac{1}{24}c_{27} = 0.$

(15) $\langle\partial^\mu\partial^\nu\partial^\nu\rangle\langle\partial^x\partial^i\partial^\alpha\rangle\langle\partial^j\partial^\mu\rangle_1\langle\partial^\alpha\rangle_1:$ $-c_3 - \dfrac{1}{240}c_6 + c_{14} - c_{15} = 0.$

(16)
$\langle\partial^j\partial^\nu\partial^\nu\rangle\langle\partial^i\partial^\alpha\partial^\beta\rangle\langle\partial^x\partial^\alpha\rangle_1\langle\partial^\beta\rangle_1:$ $\dfrac{4}{5}c_3 + c_{14} + c_{15} - c_{18} + \dfrac{1}{12}c_{28} = 0.$

(17) $\langle\partial^j\partial^\nu\partial^\nu\rangle\langle\partial^x\partial^i\partial^\alpha\rangle\langle\partial^\alpha\partial^\beta\rangle_1\langle\partial^\beta\rangle_1:$ $-\dfrac{4}{5}c_3 + c_{15} - c_{17} + \dfrac{1}{24}c_{27} = 0.$

(18) $\langle\partial^x\partial^j\partial^\mu\rangle\langle\partial^\mu\rangle_1\langle\partial^i\partial^\alpha\partial^\beta\rangle\langle\partial^\alpha\partial^\beta\rangle_1:$ $\dfrac{1}{10}c_6 + 2c_{16} + c_{22} - c_{23} = 0.$

(19) $\langle\partial^x\partial^\mu\partial^j\rangle\langle\partial^i\partial^\alpha\partial^\beta\rangle\langle\partial^\alpha\rangle_1\langle\partial^\beta\rangle_1\langle\partial^\mu\rangle_1:$ $\dfrac{7}{10}c_6 + c_{27} + c_{28} - 3c_{29} = 0.$

(20)
$\langle\partial^j\partial^\mu\partial^\nu\partial^\alpha\rangle\langle\partial^\mu\partial^\nu\partial^\alpha\rangle\langle\partial^i\partial^x\rangle_1 = \langle\partial^j\partial^\mu\partial^\alpha\partial^\alpha\rangle\langle\partial^\mu\partial^\nu\partial^\nu\rangle\langle\partial^i\partial^x\rangle_1:$
$- c_8 + c_9 + \dfrac{1}{24}c_{14} - c_{21} = 0.$

(21) $\langle\partial^j\partial^\mu\partial^\alpha\rangle\langle\partial^\mu\partial^\nu\partial^\nu\partial^\alpha\rangle\langle\partial^i\partial^x\rangle_1:$ $2c_9 - c_{12} = 0.$

(22) $\langle\partial^j\partial^\nu\partial^\nu\rangle\langle\partial^x\partial^\alpha\partial^\alpha\partial^\beta\rangle\langle\partial^i\partial^\beta\rangle_1:$ $\dfrac{1}{48}c_3 - c_7 + \dfrac{1}{24}c_{17} = 0.$

(23) $\langle\partial^x\partial^i\partial^\alpha\partial^\alpha\rangle\langle\partial^\mu\partial^\nu\partial^\nu\rangle\langle\partial^j\partial^\mu\rangle_1:$ $-\dfrac{1}{24}c_3 - \dfrac{1}{240}c_4 - c_8 + \dfrac{1}{24}c_{14} = 0.$

(24)
$$\langle\partial^j\partial^\nu\partial^\nu\rangle\langle\partial^i\partial^\alpha\partial^\alpha\partial^\beta\rangle\langle\partial^x\partial^\beta\rangle_1: \quad \frac{1}{48}c_3 + c_8 - 2c_9 + \frac{1}{24}c_{14} + \frac{1}{24}c_{18} = 0.$$

(25) $\quad \langle\partial^j\partial^\nu\partial^\nu\rangle\langle\partial^x\partial^i\partial^\alpha\partial^\beta\rangle\langle\partial^\alpha\partial^\beta\rangle_1: \quad \frac{7}{30}c_3 + 2c_8 - c_{13} + \frac{1}{24}c_{23} = 0.$

(26) $\quad \langle\partial^j\partial^x\partial^\mu\partial^\mu\rangle\langle\partial^i\partial^\alpha\partial^\beta\rangle\langle\partial^\alpha\partial^\beta\rangle_1: \quad \frac{1}{10}c_4 + 2c_7 - c_{13} + \frac{1}{24}c_{22} = 0.$

(27) $\quad \langle\partial^j\partial^\nu\partial^\nu\rangle\langle\partial^x\partial^i\partial^\alpha\partial^\beta\rangle\langle\partial^\alpha\rangle_1\langle\partial^\beta\rangle_1: \quad \frac{13}{10}c_3 + c_{15} - c_{19} + \frac{1}{8}c_{29} = 0.$

(28) $\quad \langle\partial^j\partial^x\partial^\mu\partial^\mu\rangle\langle\partial^i\partial^\alpha\partial^\beta\rangle\langle\partial^\alpha\rangle_1\langle\partial^\beta\rangle_1: \quad \frac{7}{10}c_4 + c_{17} - c_{19} + \frac{1}{24}c_{28} = 0.$

(29)
$$\langle\partial^j\partial^\nu\partial^\nu\rangle\langle\partial^x\partial^i\partial^\alpha\partial^\alpha\partial^\beta\rangle\langle\partial^\beta\rangle_1: \quad \frac{23}{240}c_3 + c_8 - 2c_{10} + \frac{1}{24}c_{15} + \frac{1}{12}c_{19} = 0.$$

(30)
$$\langle\partial^j\partial^x\partial^\mu\partial^\mu\rangle\langle\partial^i\partial^\alpha\partial^\alpha\partial^\beta\rangle\langle\partial^\beta\rangle_1: \quad \frac{13}{240}c_4 + c_7 - 2c_{10} + \frac{1}{24}c_{17} + \frac{1}{24}c_{18} = 0.$$

(31) $\quad \langle\partial^i\partial^\alpha\partial^\alpha\partial^\beta\partial^\beta\rangle\langle\partial^x\partial^j\partial^\mu\rangle\langle\partial^\mu\rangle_1: \quad \frac{1}{960}c_6 + c_9 - c_{10} + \frac{1}{24}c_{16} = 0.$

(32)
$$\langle\partial^x\partial^\mu\partial^j\rangle\langle\partial^i\partial^\alpha\partial^\alpha\partial^\nu\rangle\langle\partial^\mu\rangle_1\langle\partial^\nu\rangle_1: \quad \frac{13}{240}c_6 + c_{16} + c_{18} - 2c_{19} + \frac{1}{24}c_{27} = 0.$$

(33) $\quad \langle\partial^j\partial^\nu\partial^\nu\rangle\langle\partial^x\partial^i\partial^\alpha\partial^\alpha\partial^\mu\partial^\mu\rangle: \quad \frac{1}{576}c_3 + \frac{1}{24}c_8 + \frac{1}{24}c_{10} - 3c_{30} = 0.$

(34) $\quad \langle\partial^j\partial^x\partial^\mu\partial^\mu\rangle\langle\partial^i\partial^\alpha\partial^\alpha\partial^\nu\partial^\nu\rangle: \quad \frac{1}{960}c_4 + \frac{1}{24}c_7 + \frac{1}{24}c_9 - 3c_{30} = 0.$

There are several terms of the form $*\langle\partial^i\rangle_1$. If we remove the $\langle\partial^i\rangle_1$, they become terms in $\overline{\mathcal{M}}_{2,2}$ of codimension 3, and there is a relation between them which we can find by using Getzler's relation (with ψ^2

on x and ψ on j). The relation is

$$
\begin{aligned}
0 \;=\; & -\frac{3}{40}\langle\partial^x\partial^\mu\partial^\alpha\partial^\alpha\rangle\langle\partial^y\partial^\mu\partial^\nu\rangle\langle\partial^\nu\rangle_1 + \frac{3}{40}\langle\partial^x\partial^\mu\partial^\nu\partial^\alpha\rangle\langle\partial^\alpha\partial^y\partial^\mu\rangle\langle\partial^\nu\rangle_1 \\
& -\frac{7}{120}\langle\partial^x\partial^\mu\partial^\alpha\rangle\langle\partial^\alpha\partial^y\partial^\mu\partial^\nu\rangle\langle\partial^\nu\rangle_1 + \frac{7}{120}\langle\partial^x\partial^\nu\partial^\alpha\rangle\langle\partial^\alpha\partial^y\partial^\mu\partial^\mu\rangle\langle\partial^\nu\rangle_1 \\
& +\frac{1}{120}\langle\partial^x\partial^\mu\partial^\alpha\rangle\langle\partial^\alpha\partial^\mu\partial^\nu\rangle\langle\partial^y\partial^\nu\rangle_1 - \frac{1}{120}\langle\partial^x\partial^y\partial^\nu\partial^\alpha\rangle\langle\partial^\alpha\rangle_1\langle\partial^\mu\partial^\mu\partial^\nu\rangle \\
& -\frac{1}{120}\langle\partial^x\partial^y\partial^\alpha\rangle\langle\partial^\nu\partial^\alpha\rangle_1\langle\partial^\mu\partial^\mu\partial^\nu\rangle + \frac{1}{120}\langle\partial^y\partial^\mu\partial^\alpha\rangle\langle\partial^x\partial^\alpha\rangle_1\langle\partial^\mu\partial^\nu\partial^\nu\rangle \\
& +\text{other terms with all vertices of genus 0.}
\end{aligned}
$$

We are going to solve this relation for the term

$$\langle\partial^x\partial^\mu\partial^\alpha\rangle\langle\partial^\alpha\partial^\mu\partial^\nu\rangle\langle\partial^j\partial^\nu\rangle_1\langle\partial^i\rangle_1$$

and find an equation for all of the other terms. Among those, seven of them are of the form $*\langle\partial^i\rangle_1\langle\partial^\mu\rangle_1$ and are related by WDVV. They can be written in terms of the following 4 independent vectors.

$$\langle\partial^\alpha\partial^\alpha\partial^j\partial^\nu\rangle\langle\partial^\nu\partial^x\partial^\mu\rangle\langle\partial^i\rangle_1\langle\partial^\mu\rangle_1 :$$
(35)
$$-\frac{3}{20}c_1 - c_2 - 2c_3 - 2c_5 - \frac{1}{24}c_6 - c_{16} + 2c_{19} - c_{24} + \frac{1}{24}c_{27} = 0.$$

(36) $\qquad \langle\partial^\alpha\partial^\alpha\partial^x\partial^\nu\rangle\langle\partial^\nu\partial^j\partial^\mu\rangle\langle\partial^i\rangle_1\langle\partial^\mu\rangle_1 : \quad -c_2 - c_4 = 0.$

(37)
$$\langle\partial^\alpha\partial^\alpha\partial^\mu\partial^\nu\rangle\langle\partial^\nu\partial^x\partial^j\rangle\langle\partial^i\rangle_1\langle\partial^\mu\rangle_1 : \quad \frac{11}{240}c_1 + c_2 + 2c_3 + c_5 - c_{18} + c_{24} = 0.$$

(38)
$$\langle\partial^\alpha\partial^j\partial^\nu\rangle\langle\partial^\nu\partial^x\partial^\mu\partial^\alpha\rangle\langle\partial^i\rangle_1\langle\partial^\mu\rangle_1 : \quad -\frac{1}{10}c_1 - 2c_3 - c_5 + 4c_{19} - c_{23} - c_{24} = 0.$$

There are additional 4 independent vectors
(39)
$$\langle\partial^x\partial^\beta\partial^\gamma\rangle\langle\partial^\gamma\partial^j\partial^\alpha\rangle\langle\partial^\alpha\partial^\beta\rangle_1\langle\partial^i\rangle_1 : \quad \frac{1}{10}c_1 + c_5 + c_{22} - c_{23} - 2c_{25} + 2c_{26} = 0.$$

(40) $\langle\partial^x\partial^\beta\partial^\mu\rangle\langle\partial^\mu\partial^j\partial^\alpha\rangle\langle\partial^\alpha\rangle_1\langle\partial^\beta\rangle_1\langle\partial^i\rangle_1 : \quad -\frac{7}{10}c_1 - c_6 - c_{28} + 3c_{29} = 0.$

(41)
$$\langle\partial^\mu\partial^\nu\partial^\nu\rangle\langle\partial^i\partial^\mu\partial^\alpha\rangle\langle\partial^x\partial^\alpha\rangle_1\langle\partial^j\rangle_1 : \quad \frac{1}{20}c_1 + c_2 + c_3 + c_5 - c_{14} + c_{15} + c_{18} - c_{22} = 0.$$

(42)
$$\langle\partial^\alpha\partial^\beta\partial^\beta\rangle\langle\partial^x\partial^j\partial^\gamma\rangle\langle\partial^\alpha\partial^\gamma\rangle_1\langle\partial^i\rangle_1 : \quad -\frac{11}{240}c_1 - c_2 - 2c_3 - c_5 - c_{14} + c_{15} = 0.$$

The remaining connected strata are all of codimension 3 in $\overline{\mathcal{M}}_{2,3}$. There are four induced equations. First of all, by taking Getzler's relation in $\overline{\mathcal{M}}_{1,4}$, adding another marked point, and then identifying either two of the first four marked points or the fifth marked point with one of the others. Secondly, by taking the Belorousski-Pandharipande relation, adding ψ at the marked point x, and then simplifying. The relations we obtain are the following:

$$
\begin{aligned}
0 = {} & \langle \partial^x \partial^i \partial^\mu \rangle \langle \partial^j \partial^\alpha \partial^\nu \rangle \langle \partial^\alpha \partial^\mu \partial^\nu \rangle_1 + \langle \partial^x \partial^j \partial^\mu \rangle \langle \partial^i \partial^\alpha \partial^\nu \rangle \langle \partial^\alpha \partial^\mu \partial^\nu \rangle_1 \\
& + \langle \partial^x \partial^\alpha \partial^\mu \rangle \langle \partial^i \partial^j \partial^\nu \rangle \langle \partial^\alpha \partial^\mu \partial^\nu \rangle_1 - \langle \partial^x \partial^i \partial^\mu \rangle \langle \partial^j \partial^\mu \partial^\nu \rangle \langle \partial^\alpha \partial^\alpha \partial^\nu \rangle_1 \\
& - \langle \partial^x \partial^i \partial^\mu \rangle \langle \partial^\alpha \partial^\mu \partial^\nu \rangle \langle \partial^j \partial^\alpha \partial^\nu \rangle_1 - \langle \partial^x \partial^j \partial^\mu \rangle \langle \partial^\alpha \partial^\mu \partial^\nu \rangle \langle \partial^i \partial^\alpha \partial^\nu \rangle_1 \\
& - \langle \partial^i \partial^j \partial^\mu \rangle \langle \partial^\alpha \partial^\mu \partial^\nu \rangle \langle \partial^x \partial^\alpha \partial^\nu \rangle_1 + \text{other terms},
\end{aligned}
$$

$$
\begin{aligned}
0 = {} & \langle \partial^x \partial^i \partial^\mu \rangle \langle \partial^\alpha \partial^\alpha \partial^\nu \rangle \langle \partial^j \partial^\mu \partial^\nu \rangle_1 - \langle \partial^x \partial^\alpha \partial^\mu \rangle \langle \partial^\alpha \partial^\mu \partial^\nu \rangle \langle \partial^i \partial^j \partial^\nu \rangle_1 \\
& - \langle \partial^i \partial^\alpha \partial^\mu \rangle \langle \partial^\alpha \partial^\mu \partial^\nu \rangle \langle \partial^x \partial^j \partial^\nu \rangle_1 + \text{other terms},
\end{aligned}
$$

$$
\begin{aligned}
0 = {} & \langle \partial^i \partial^j \partial^\mu \rangle \langle \partial^\alpha \partial^\alpha \partial^\nu \rangle \langle \partial^x \partial^\mu \partial^\nu \rangle_1 - \langle \partial^i \partial^\alpha \partial^\mu \rangle \langle \partial^\alpha \partial^\mu \partial^\nu \rangle \langle \partial^j \partial^x \partial^\nu \rangle_1 \\
& - \langle \partial^j \partial^\alpha \partial^\mu \rangle \langle \partial^\alpha \partial^\mu \partial^\nu \rangle \langle \partial^i \partial^x \partial^\nu \rangle_1 + \text{other terms},
\end{aligned}
$$

$$
\begin{aligned}
0 = {} & -\langle \partial^\mu \partial^\alpha \partial^\nu \rangle \langle \partial^\mu \partial^i \partial^j \rangle \langle \partial^\nu \rangle_1 \langle \partial^x \partial^\alpha \rangle_1 - \langle \partial^x \partial^j \partial^\alpha \rangle \langle \partial^\alpha \partial^\mu \partial^\nu \rangle \langle \partial^\mu \rangle_1 \langle \partial^\nu \partial^i \rangle_1 \\
& - \langle \partial^x \partial^i \partial^\alpha \rangle \langle \partial^\alpha \partial^\mu \partial^\nu \rangle \langle \partial^\mu \rangle_1 \langle \partial^\nu \partial^j \rangle_1 - \langle \partial^x \partial^\mu \partial^\alpha \rangle \langle \partial^\mu \partial^i \partial^j \rangle \langle \partial^\alpha \partial^\nu \rangle_1 \langle \partial^\nu \rangle_1 \\
& + \langle \partial^x \partial^\alpha \partial^\nu \rangle \langle \partial^\mu \partial^i \partial^j \rangle \langle \partial^\alpha \partial^\mu \rangle_1 \langle \partial^\nu \rangle_1 + \langle \partial^x \partial^j \partial^\alpha \rangle \langle \partial^i \partial^\mu \partial^\nu \rangle \langle \partial^\mu \rangle_1 \langle \partial^\nu \partial^\alpha \rangle_1 \\
& + \langle \partial^x \partial^i \partial^\alpha \rangle \langle \partial^j \partial^\mu \partial^\nu \rangle \langle \partial^\mu \rangle_1 \langle \partial^\nu \partial^\alpha \rangle_1 + \text{other terms}.
\end{aligned}
$$

After solving the four relations above for the following four terms

$$
\langle \partial^x \partial^i \partial^\mu \rangle \langle \partial^j \partial^\mu \partial^\nu \rangle \langle \partial^\alpha \partial^\alpha \partial^\nu \rangle_1,
$$
$$
\langle \partial^x \partial^i \partial^\mu \rangle \langle \partial^\alpha \partial^\alpha \partial^\nu \rangle \langle \partial^j \partial^\mu \partial^\nu \rangle_1,
$$
$$
\langle \partial^i \partial^j \partial^\mu \rangle \langle \partial^\alpha \partial^\alpha \partial^\nu \rangle \langle \partial^x \partial^\mu \partial^\nu \rangle_1,
$$
$$
\langle \partial^x \partial^\mu \partial^\alpha \rangle \langle \partial^\mu \partial^i \partial^j \rangle \langle \partial^\alpha \partial^\nu \rangle_1 \langle \partial^\nu \rangle_1,
$$

we obtain the following equations for the other terms.

(43) $\qquad \langle \partial^x \partial^\alpha \partial^\beta \rangle \langle \partial^\beta \partial^i \partial^j \rangle \langle \partial^\alpha_1 \rangle_2 : \qquad -\dfrac{1}{2}c_1 + 4c_5 - \dfrac{1}{2}c_6 = 0$

(44)
$\langle \partial^\alpha \partial^\beta \partial^\beta \rangle \langle \partial^x \partial^\alpha \partial^\gamma \rangle \langle \partial^i \partial^j \partial^\gamma \rangle_1 : \qquad -\dfrac{1}{40}c_1 - \dfrac{1}{2}c_2 - \dfrac{1}{2}c_3 - \dfrac{1}{2}c_5 + 2c_8 - \dfrac{1}{2}c_{15} = 0.$

(45)
$\langle \partial^\mu \partial^\nu \partial^\nu \rangle \langle \partial^i \partial^\mu \partial^\alpha \rangle \langle \partial^j \partial^x \partial^\alpha \rangle_1 : \qquad \dfrac{1}{240}c_1 - c_3 - \dfrac{1}{60}c_5 + 4c_8 + 2c_{12} - c_{15} - 3c_{21} = 0.$

(46)

$$\langle\partial^x\partial^\beta\partial^\gamma\rangle\langle\partial^\gamma\partial^i\partial^\alpha\rangle\langle\partial^j\partial^\alpha\partial^\beta\rangle_1 : \quad -\frac{1}{10}c_1 - \frac{29}{30}c_5 - 4c_{11} + c_{16} + 3c_{20} - 3c_{21} = 0.$$

(47)

$$\langle\partial^i\partial^j\partial^\beta\rangle\langle\partial^\beta\partial^\mu\partial^\nu\rangle\langle\partial^x\partial^\mu\partial^\nu\rangle_1 : \quad \frac{1}{30}c_5 - 2c_{11} + \frac{1}{2}c_{16} + 6c_{21} - \frac{1}{2}c_{24} = 0.$$

(48) $\langle\partial^x\partial^j\partial^\nu\rangle\langle\partial^i\partial^\alpha\partial^\beta\rangle\langle\partial^\nu\partial^\alpha\partial^\beta\rangle_1 : \quad \frac{1}{15}c_5 + 8c_{11} - c_{16} - 3c_{20} + 3c_{21} = 0.$

(49) $\langle\partial^x\partial^\mu\partial^\nu\rangle\langle\partial^i\partial^j\partial^\beta\rangle\langle\partial^\mu\partial^\nu\partial^\beta\rangle_1 : \quad -\frac{1}{30}c_5 + 4c_{11} - \frac{1}{2}c_{16} - \frac{1}{2}c_{25} = 0.$

(50)
$$\langle\partial^x\partial^\beta\partial^\gamma\rangle\langle\partial^\gamma\partial^i\partial^\alpha\rangle\langle\partial^j\partial^\alpha\rangle_1\langle\partial^\beta\rangle_1 :$$
$$-\frac{7}{5}c_1 + \frac{7}{5}c_5 - c_6 + 2c_{23} - 2c_{24} - 6c_{25} + 2c_{26} + c_{27} = 0.$$

(51)

$$\langle\partial^i\partial^j\partial^\beta\rangle\langle\partial^\beta\partial^\mu\partial^\nu\rangle\langle\partial^x\partial^\mu\rangle_1 : \quad \frac{4}{5}c_5 + 2c_{22} + 2c_{24} - 2c_{25} + \frac{1}{2}c_{27} - c_{28} = 0.$$

(52)

$$\langle\partial^x\partial^\mu\partial^j\rangle\langle\partial^i\partial^\alpha\partial^\beta\rangle\langle\partial^\mu\partial^\alpha\rangle_1\langle\partial^\beta\rangle_1 : \quad \frac{3}{5}c_5 - 2c_{23} + 2c_{24} + 6c_{25} + 2c_{26} - c_{27} = 0.$$

(53) $\langle\partial^x\partial^\mu\partial^\nu\rangle\langle\partial^i\partial^j\partial^\beta\rangle\langle\partial^\nu\partial^\beta\rangle_1\langle\partial^\mu\rangle_1 : \quad -\frac{4}{5}c_5 + 2c_{16} + 2c_{25} - c_{27} = 0.$

Solving the equations (5)-(53) gives the following coefficients (let $c_1 = -1$):

$$c_2 = -\frac{5}{72} \qquad c_3 = -\frac{1}{252} \qquad c_4 = \frac{5}{72} \qquad c_5 = \frac{5}{42}$$

$$c_6 = \frac{41}{21} \qquad c_7 = -\frac{11}{40320} \qquad c_8 = -\frac{1}{13440} \qquad c_9 = -\frac{1}{8064}$$

$$c_{10} = \frac{191}{120960} \qquad c_{11} = -\frac{1}{5040} \qquad c_{12} = -\frac{1}{4032} \qquad c_{13} = \frac{17}{2880}$$

$$c_{14} = \frac{1}{840} \qquad c_{15} = -\frac{1}{336} \qquad c_{16} = -\frac{1}{126} \qquad c_{17} = -\frac{23}{5040}$$

$$c_{18} = -\frac{17}{5040} \qquad c_{19} = \frac{113}{2520} \qquad c_{20} = \frac{1}{210} \qquad c_{21} = 0$$

$$c_{22} = -\frac{1}{84} \qquad c_{23} = \frac{211}{1260} \qquad c_{24} = \frac{1}{1260} \qquad c_{25} = -\frac{1}{630}$$

$$c_{26} = \frac{11}{140} \qquad c_{27} = -\frac{4}{35} \qquad c_{28} = \frac{2}{105} \qquad c_{29} = \frac{89}{210}$$

$$c_{30} = \frac{1}{53760}$$

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