Homework 8, Math 6610, due Oct. 31

1. In the following you will be shown the steps to solve the Sylvester or Lyapunov equation $AX - XB = C$, where X and C are m by n, A is m by m and B is n by n. This is a system of mn linear equations for the entries of X.

   (a) Given the Schur decompositions of A and B, show how $AX - XB = C$ can be transformed into a similar system $A'Y - YB' = C'$ where A' and B' are upper triangular.

   (b) Show how to solve for the entries of Y one at a time by a process analogous to back substitution. What condition on the eigenvalues of A and B would guarantee that the system of equations is nonsingular?

   (c) Show how to transform Y to get the solution X.

2. Suppose $T = \begin{bmatrix} A & C \\ 0 & B \end{bmatrix}$ is in Schur form. We want to find a matrix S so that $S^{-1}TS = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. It turns out we can choose S of the form $\begin{bmatrix} I & R \\ 0 & I \end{bmatrix}$. Show how to solve for R. What's the connection between this problem and Problem 1?

3. The initial value problem

$$\begin{aligned} \dot{x}(t) &= y(t) \qquad x(0) = 1 \\ \dot{y}(t) &= -x(t) \qquad y(0) = 0 \end{aligned}$$

   has solution $x(t) = \cos(t)$ and $y(t) = \sin(t)$. Let $h > 0$. Here are three reasonable iterations that can be used to compute approximations $x_k \approx x(kh)$ and $y_k \approx y(kh)$, $k = 1, 2, 3, \ldots$, assuming that $x_0 = 1$ and $y_0 = 0$:

   - Method 1

$$\begin{aligned} x_{k+1} &= x_k + hy_k \\ y_{k+1} &= y_k - hx_k \end{aligned}$$

   - Method 2

$$\begin{aligned} x_{k+1} &= x_k + hy_k \\ y_{k+1} &= y_k - hx_{k+1} \end{aligned}$$

   - Method 3

$$\begin{aligned} x_{k+1} &= x_k + hy_{k+1} \\ y_{k+1} &= y_k - hx_{k+1} \end{aligned}$$

   Express each method in the form

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \end{bmatrix} = A_h \begin{bmatrix} x_k \\ y_k \end{bmatrix}$$

   where A_k is a 2×2 matrix. For each case, compute the eigenvalues and use them to discuss $\lim x_k$ and $\lim y_k$ as $k \to \infty$.