Homework No. 2, Math 6610-1, Due Sept. 19

1. Given the matrix
$$A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix},$$
without using any properties proved in lemma 1.7 of the textbook, find $||A||_1$, $||A||_2$ and $||A||_\infty$ based on the operator norm definitions, by explicitly obtaining the directions where the maximums of the ratio $||Ax||/||x||$ are attained.

2. Consider a real, non-singular $n \times n$ matrix A and vectors b, $x \in \mathbb{R}^n$. We are concerned with the error in solving the linear system $Ax = b$. The numerical solution $\hat{x}$ is only an approximate solution to this equation and it satisfies a perturbed equation $\hat{A}\hat{x} = b$ exactly, where $\hat{A} = A + \delta A$. Let the error be defined as $\delta x = \hat{x} - x$, we are interested in bounding $||\delta x||$ in terms of $||\delta A||||A||$, or $||\delta A||||\hat{A}||$, and $||x||$, or $||\hat{x}||$, where $||\cdot||$ is some suitable norm, and we proceed as follows:

   (a) First prove the identity
   $$A^{-1} - \hat{A}^{-1} = A^{-1} \cdot \delta A \cdot \hat{A}^{-1},$$
   and hence deduce that
   $$||A^{-1} - \hat{A}^{-1}|| \leq ||A^{-1}|| \cdot ||\delta A|| \cdot ||\hat{A}^{-1}||.$$

   (b) Assuming that $\hat{\delta} = ||\delta A||||\hat{A}^{-1}|| < 1$, show that
   $$||A^{-1}|| \leq \frac{1}{1-\hat{\delta}}||\hat{A}^{-1}||, \qquad ||A^{-1} - \hat{A}^{-1}|| \leq \frac{\hat{\delta}}{1-\hat{\delta}}||\hat{A}^{-1}||.$$

   (c) By comparing the equations $Ax = b$ and $(A + \delta A)(x + \delta x) = b$, show that
   $$||\delta x|| \leq \frac{\hat{\delta}}{1-\hat{\delta}}||\hat{x}||.$$
   Similarly, show that if $\delta = ||\delta A||||A^{-1}|| < 1$,
   $$||\delta x|| \leq \frac{\delta}{1-\delta}||x||.$$

3. Let $A = LU$ be the LU factorization of n-by-n A with $|l_{ij}| \leq 1$. Let a_i^T and u_i^T denote the ith rows of A and U, respectively. Verify the equation
$$u_i^T = a_i^T - \sum_{j=1}^{i-1} l_{ij} u_j^T$$
and use it to show that $||U||_\infty \leq 2^{n-1}||A||_\infty$.

4. A matrix is strictly diagonally dominant if

$$|a_{i,i}| > \sum_{j=1,j\neq i}^{n} |a_{i,j}| \qquad i = 1, ..., n$$

Show that there is no pivoting required for a strictly diagonally dominant matrix in a LU decomposition. In other words, after one step of the Gaussian elimination, the remaining submatrix is still strictly diagonally dominant.

5. The system $Ax = b$ where

$$A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 10^{-10} & 10^{-10} \\ 1 & 10^{-10} & 10^{-10} \end{bmatrix} \quad b = \begin{bmatrix} 2(1+10^{-10}) \\ -10^{-10} \\ 10^{-10} \end{bmatrix}$$

has solution $x = (10^{-10}, -1, 1)^T$. (a) Show that if $(A + E)y = b$ and $|E| \leq 10^{-8}|A|$, then $|x - y| \leq 10^{-7}|x|$. That is, small relative changes in A's entries do not induce large changes in x even though $\kappa_\infty(A) = 10^{10}$. (b) Define $D = diag(10^{-5}, 10^5, 10^5)$. Show $\kappa_\infty(DAD) \leq 5$.