# Math 6610 Take Home Final, due Dec. 12, 5:00 pm

1. (20 points) Here is another form of the error bound for a linear system in terms of the condition number. Suppose the linear system

$$A\mathbf{x} = \mathbf{b}$$

is perturbed to

$$(A + \delta A)(\mathbf{x} + \delta \mathbf{x}) = \mathbf{b} + \delta \mathbf{b}$$

and we are interested in a bound for $\delta \mathbf{x}$. Assume $A$ is $n \times n$ and nonsingular, and the perturbations are small so that

$$||\delta A|| \le \epsilon ||A||, \quad ||\delta \mathbf{b}|| \le \epsilon ||\mathbf{b}||$$

for some small $\epsilon > 0$, and $\epsilon \kappa(A) = r < 1$. Show

(a) $A + \delta A$ is nonsingular;

(b) If $\tilde{\mathbf{x}} = \mathbf{x} + \delta \mathbf{x}$, then

$$||\tilde{\mathbf{x}}|| \le \frac{1+r}{1-r} ||\mathbf{x}||;$$

(c)

$$\frac{||\delta \mathbf{x}||}{||\mathbf{x}||} \le \frac{2\epsilon}{1-r} \kappa(A).$$

2. (15 points) Given a $m \times n$ matrix $A$ where $m \ge n$,

(a) Show that the largest singular value

$$\sigma_{\max} = \max_{\mathbf{y} \in \mathbf{R}^m,\ \mathbf{x} \in \mathbf{R}^n} \frac{\mathbf{y}^T A \mathbf{x}}{||\mathbf{x}||_2 ||\mathbf{y}||_2}$$

(b) If we change the max to the min in the expression, do we get the smallest singular value? Explain why.

3. (20 points) For the least square problem, suppose $||A\mathbf{x} - \mathbf{b}||_2$ ($A$ is $m \times n$ where $m \ge n$) is the total error of the curve fitting. In order that the curve behaves well, sometimes we require that the oscillation be kept under control, and one way to do that is to make sure the total variation is bounded by some pre-specified constant. Let us simplify the situation by assuming that the total variation is given by $||B\mathbf{x}||_2^2$, where $B \in R^{n \times n}$ is some nonsingular matrix. We have two alternative formulations:

(a) Minimize the corrected objective function $f(\mathbf{x}) = ||A\mathbf{x} - \mathbf{b}||_2^2 + \alpha||B\mathbf{x}||_2^2$ for some constant $\alpha > 0$;

(b) Minimize $g(\mathbf{x}) = ||A\mathbf{x} - \mathbf{b}||_2$, subject to $||B\mathbf{x}||_2 \le \beta$, for some constant $\beta > 0$.

There are different ways to define total oscillation for the solution $\mathbf{x}$. Imagine that the components of $\mathbf{x}$ approximate the function values $f(x_i), i = 1, \dots n$, for some function $f(x)$ at equally spaced points $x_j$. What would you choose the matrix $B$ if your definition of the total oscillation is (i). $\int |f(x)|^2 dx$, (ii). $\int |f'(x)|^2 dx$, or (iii). $\int |f''(x)|^2$?

Use the modified normal equation for the first and SVD for the second to obtain the solutions. In the second problem, you will need to discuss different cases where the solution occurs within the feasible set ($||B\mathbf{x}||_2 < \beta$), or on the boundary of the feasible set ($||B\mathbf{x}||_2 = \beta$). In the latter case you need to use the method of Lagrange multipliers.

4. (15 points) In the inverse iteration method for symmetric eigenvalue problems, we need to solve the linear system

$$(A - \mu I)y_{i+1} = x_i$$

repeatly, and $\mu$ is chosen to be close to one of the eigenvalues of $A$. As we know that means $\kappa(A - \mu I)$ will be large and the linear system is ill-conditioned. We do not expect the solution $y$ to be accurately obtained. Will this be a big problem for the algorithm? Try to come up with an answer from the following approach: what we need is $x_{i+1} = y_{i+1}/||y_{i+1}||$. Suppose that $(A - \mu I)y = x$ is solved backward stably, yielding a computed solution $\tilde{y}$ and it may be quite off from $y$. Show that $\tilde{y}/||\tilde{y}||$ will not be far from $y/||y||$. Can you make the same statement for general non-symmetric matrices?

(Hint: By the assumed backward stability, it means there exists a symmetric matrix $\tilde{A}$ such that

$$(\tilde{A} - \mu I)\tilde{y} = x$$

and $||A - \tilde{A}||/||A|| = O(\epsilon_{\text{mach}})$. It may be helpful to expand $y, \tilde{y}$ and $x$ in respective eigenspaces of $A$ and $\tilde{A}$ and use the stability results for symmetric matrices.)

5. (15 points) For any $n$-by-$n$ matrix $A$, show that the spectral radius $\rho(A) < 1$ if and only if $\lim_{i\to\infty} A^i = 0$. Let $S_n = \sum_{i=0}^n A^i$. Show that $\lim_{n\to\infty} S_n$ exists if and only if $\rho(A) < 1$, and then

$$\lim_{n\to\infty} S_n = (I - A)^{-1}$$

6. (15 points) Given two vectors $f, g \in C^N$, the convolution $f * g$ of $f$ and $g$ is a vector with components

$$(f * g)_k = \sum_{j=0}^{N-1} f_j g_{[k-j]}, \qquad k = 0, 1, \dots, N-1.$$

Here

$$[k-j] = \begin{cases} k-j, & k \geq j \\ k-j+N. & k < j \end{cases}$$

Show that the DFT coefficients are related by

$$(\widehat{f * g})_k = \hat{f}_k \cdot \hat{g}_k, \quad k = 0, 1, \ldots, N-1.$$